

Durham Research Online

Deposited in DRO:

21 June 2016

Version of attached file:

Published Version

Peer-review status of attached file:

Peer-reviewed

Citation for published item:

Banerjee, Nabamita and Dutta, Suvankar and Jain, Akash (2016) 'Null fluids : a new viewpoint of Galilean fluids.', Physical review D., 93 (10). p. 105020.

Further information on publisher's website:

<http://dx.doi.org/10.1103/PhysRevD.93.105020>

Publisher's copyright statement:

Reprinted with permission from the American Physical Society: Physical Review D 93, 105020 © (2016) by the American Physical Society. Readers may view, browse, and/or download material for temporary copying purposes only, provided these uses are for noncommercial personal purposes. Except as provided by law, this material may not be further reproduced, distributed, transmitted, modified, adapted, performed, displayed, published, or sold in whole or part, without prior written permission from the American Physical Society.

Additional information:

Use policy

The full-text may be used and/or reproduced, and given to third parties in any format or medium, without prior permission or charge, for personal research or study, educational, or not-for-profit purposes provided that:

- a full bibliographic reference is made to the original source
- a [link](#) is made to the metadata record in DRO
- the full-text is not changed in any way

The full-text must not be sold in any format or medium without the formal permission of the copyright holders.

Please consult the [full DRO policy](#) for further details.

Null fluids: A new viewpoint of Galilean fluidsNabamita Banerjee,^{1,*} Suvankar Dutta,^{2,†} and Akash Jain^{3,‡}¹*Department of Physics, Indian Institute of Science Education and Research (IISER),
Pune, Maharashtra 411008, India*²*Department of Physics, Indian Institute of Science Education and Research (IISER),
Bhopal, Madhya Pradesh 462030, India*³*Department of Mathematical Sciences & Centre for Particle Theory,
Durham University, Durham DH13LE, United Kingdom*

(Received 12 January 2016; published 13 May 2016)

In this article, we study a Galilean fluid with a conserved $U(1)$ current up to anomalies. We construct a relativistic system, which we call a *null fluid* and show that it is in one-to-one correspondence with a Galilean fluid living in one lower dimension. The correspondence is based on light cone reduction, which is known to reduce the Poincaré symmetry of a theory to Galilean in one lower dimension. We show that the proposed *null fluid* and the corresponding Galilean fluid have exactly same symmetries, thermodynamics, constitutive relations, and equilibrium partition to all orders in the derivative expansion. We also devise a mechanism to introduce $U(1)$ anomaly in even dimensional Galilean theories using light cone reduction, and study its effect on the constitutive relations of a Galilean fluid.

DOI: [10.1103/PhysRevD.93.105020](https://doi.org/10.1103/PhysRevD.93.105020)**I. INTRODUCTION AND SUMMARY**

Nonrelativistic systems enjoy an active interest in the physics community primarily for two main reasons. First, they are expected to be realized in the low energy physics experiments. A second and more fundamentally grounded reason is that a nonrelativistic system can be thought of as an effective low energy description of an underlying relativistic theory. Hence, it is natural to expect that the constitutive relations of a nonrelativistic fluid, obtained as an effective description of a relativistic theory, may contain new terms which are not considered in the coarse grained description of hydrodynamics [1]. For example, if the system breaks parity symmetry at the microscopic level, it will force us to add parity-odd terms in the constitutive relations. The goal of this paper is to revisit the paradigm of nonrelativistic charged hydrodynamics. We devise a consistent mechanism to derive the parity-even and odd terms in the constitutive relations of a nonrelativistic fluid up to leading order in derivatives, starting from a relativistic theory.

Galilean fluids¹ have been an interesting and active topic of research in the recent years [2–10]. Authors in [2] worked out a consistent way to write the Galilean fluid constitutive relations in Newton-Cartan covariant formalism, and used the second law of thermodynamics to constrain the hydrodynamic transport in two spatial dimensions. It is known that

Newton-Cartan geometry with Galilean isometry follows from light cone reduction of a relativistic geometry in one higher dimension [11,12]. The idea behind this is that the Poincaré algebra in $(d+2)$ -dim has a $(d+1)$ -dim Galilean subalgebra² embedded into it. As suggested in [5,13], this approach can be used to construct Galilean covariant tensors in the Newton-Cartan formalism, which is otherwise a nontrivial task. Similar ideas were also used in [2] where authors constructed an extended representation of the Galilean group by embedding it into one higher dimension, and used it to present the Galilean hydrodynamics in a manifestly covariant manner.

In this work, we take this approach a step ahead and ask if we can construct a relativistic fluid, whose symmetry algebra when restricted to the Galilean subalgebra, is equivalent to a Galilean fluid in one lower dimension. This idea has been explored in the past, starting with [14] which showed that dynamics of a relativistic fluid reduces to that of a Galilean fluid under light cone reduction. However in [15], we observed that this naïve approach runs into some troubles—the thermodynamics that the reduced Galilean fluid follows is restricted (because mass chemical potential is not an independent variable after reduction; see footnote 9 for related comments). We also found that the parity-odd sector only survives if the fluid is incompressible and is kept in a constant magnetic field. It strongly hints that to obtain the most generic Galilean fluid via light cone reduction, we need to start with a modified relativistic system.

More precisely, we start with a flat background (metric and gauge field),

*nabamita@iiserpune.ac.in†suvankar@iiserb.ac.in‡akash.jain@durham.ac.uk

¹A nonrelativistic system is defined by $c \rightarrow \infty$ limit of a relativistic system, while a Galilean system is one whose isometry group is Galilean. Hence, not every Galilean system needs to be nonrelativistic. In this paper however, we only talk about the Galilean theories, as they are much easier to handle.

²To be more precise, what we call a Galilean group (algebra) is generally known as the *Bargmann group (algebra)*, which is a *central extension of the Galilean group*.

$$ds_{\text{flat}}^2 = -2dx^- dt + \sum_{i=1}^d (dx^i)^2, \quad \mathcal{A}_{\text{flat}} = 0, \quad (1)$$

which has $(d+2)$ -dim Poincaré invariance. $(d+1)$ -dim Galilean algebra sits inside Poincaré—all generators which commute with $P_- = \partial_-$ (cf. [3]). Hence a theory on this background which respects x^- independent isometries $x^M \rightarrow x^M + \xi^M(t, \vec{x})$, $x^M = \{x^-, t, x^i\}$ enjoys Galilean invariance. Compactifying the x^- direction, we can recover a $(d+1)$ -dim flat Galilean background on which Galilean theories can be defined. This is known as the *light cone reduction (LCR)*. We turn on x^- independent fluctuations around the flat background,

$$ds^2 = -2e^{-\Phi}(dt + a_i dx^i)(dx^- - \mathcal{B}_t dt - \mathcal{B}_i dx^i) + g_{ij} dx^i dx^j, \\ \mathcal{A} = \mathcal{A}_t dt + \mathcal{A}_i dx^i. \quad (2)$$

Galilean theories can then be described by a partition function $\mathcal{Z}[\mathcal{B}_t, \mathcal{B}_i, \Phi, a_i, g_{ij}, \mathcal{A}_t, \mathcal{A}_i]$. Treating these fluctuations as sources, we can define the following observables, evaluated in the absence of sources (i.e. on the flat background) [5,16],

$$\begin{aligned} \rho &= \left. \frac{\delta W}{\delta \mathcal{B}_t} \right|_{\text{flat}}, & j_\rho^i &= \left. \frac{\delta W}{\delta \mathcal{B}_i} \right|_{\text{flat}}, & \epsilon &= \left. \frac{\delta W}{\delta \Phi} \right|_{\text{flat}}, \\ j_\epsilon^i &= \left. \frac{\delta W}{\delta a_i} \right|_{\text{flat}}, & t^{ij} &= 2 \left. \frac{\delta W}{\delta g_{ij}} \right|_{\text{flat}}, \\ q &= \left. \frac{\delta W}{\delta \mathcal{A}_t} \right|_{\text{flat}}, & j_q^i &= \left. \frac{\delta W}{\delta \mathcal{A}_i} \right|_{\text{flat}}. \end{aligned} \quad (3)$$

Here $W = \ln \mathcal{Z}$, and $\rho, j_\rho^i, \epsilon, j_\epsilon^i, t^{ij}, q, j_q^i$ are mass density, mass current, energy density, energy current, stress tensor, charge density and charge current respectively of the Galilean theory. Invariance of the partition function under x^- independent diffeomorphisms and $U(1)$ gauge transformations will imply the following conservation equations,

$$\begin{aligned} \partial_t \rho + \partial_i j_\rho^i &= 0, & \partial_t \epsilon + \partial_i j_\epsilon^i &= 0, \\ \partial_t j_\rho^i + \partial_j t^{ji} &= 0, & \partial_t q + \partial_i j_q^i &= 0. \end{aligned} \quad (4)$$

These are exactly what we expect for a Galilean system, if we identify t with the Galilean time, as suggested by the notation.

The above procedure can be made manifestly covariant as proposed in [11] and later developed in [12,13,17] and many others. Consider a curved background: a metric G_{MN} , the associated Levi-Civita connection Γ_{NR}^M (with associated covariant derivative ∇_M) and a gauge field \mathcal{A}_M along with a covariantly constant null isometry V^M , i.e. $\nabla_M V^N$, $V^M V_M = 0$, $\mathcal{L}_V G_{MN} = \mathcal{L}_V \mathcal{A}_M = 0$. This is a natural generalization of the *Bargmann structures* [11] (spacetimes

with a covariantly constant null Killing vector) to backgrounds with a gauge field. We define a *null background*³ to be this generalized Bargmann structure which further satisfies $V^M \mathcal{A}_M = 0$. Theories on a null background, which we call *null theories*, are demanded to be invariant under V^M preserving diffeomorphisms and gauge transformations. The background given in Eq. (2) with respective Galilean symmetry then just follows by a choice of basis $x^M = \{x^-, t, x^i\}$ such that $V = \partial_-$ (t is not necessarily null). It suggests that null theories are entirely equivalent to Galilean theories, and are related by merely this choice of basis. More formally, null theories exhibit Galilean invariance upon null reduction, i.e. getting rid of V direction through compactification.

We can now study a *null fluid* on this null background, with the hope to get the most generic Galilean fluid after reduction. Unlike “usual” relativistic fluids, in this case the isometry V is also a background field and hence must be considered while writing the respective constitutive relations. This simple consideration happens to resolve all the issues we enlisted before. In fact it does much more than; even before LCR, $(d+2)$ -dim null fluid is essentially equivalent to a $(d+1)$ -dim Galilean fluid, as they have same symmetries. As we shall show, their constitutive relations, conservation equations, thermodynamics etc. match exactly to all orders in the derivative expansion.

Another motivation to study null backgrounds is Galilean anomalies.⁴ The thumb rule for anomalies tells us that they can only exist in even dimensions. But since light cone reduction reduces dimension of the theory by one, even if we start with an even-dimensional anomalous relativistic theory, the reduced Galilean theory is odd-dimensional and hence all anomalous terms should vanish.⁵ However this argument about dimensionality can be bypassed by working with the null backgrounds. Since

³Here, the definition of null backgrounds has been adapted to a torsionless spacetime. When the connection is not torsionless, the isometry V is demanded to be covariantly constant with respect to the torsional-connection as opposed to the Levi-Civita connection, and further conditions need to be added to this definition (V^M acts as an isometry on the torsion tensor and the component of the connection along V is fixed to be $V^M \Gamma_{MS}^R = -\partial_S V^R$). See [18] for an extensive discussion on the torsional case.

⁴We will only be talking about global t’Hooft anomalies appearing in Galilean theories as described by [19]. Our working definition of anomaly shall be that the respective conservation laws are violated by certain terms purely dependent on the background sources. We do not dwell in the microscopic interpretation of these anomalies.

⁵This is in contrast with the results of [19], where the author found that the relativistic anomalies survive the light cone reduction and show up as gauge/gravitational and Milne anomalies in the Galilean theory. We observe that this is because of the presence of extra scalar sources in the Galilean theory that are reminiscent of reduction and must be switched off in a physically realizable theory. We present a detailed analysis on these issues in a companion paper [18].

there is an extra vector field V in the theory, the tensor structure allows for anomalies only in odd dimensions. In fact, one can reconstruct anomalies in an even dimensional Galilean theory by starting with an odd dimensional anomalous null theory. We would like to use the null fluid construction to see how these anomalies affect the Galilean hydrodynamic transport. At this point, we do not have a clear understanding of the field theoretic interpretation of these anomalous terms. There has been some work in this direction in [20] where authors discussed chiral anomalies for Lifshitz fermions using path integral methods. It is interesting to note that the form of anomalies found by these authors exactly match with our proposal. However, more investigation is needed to understand the physical origin of these anomalies. This is beyond the scope of this paper.

It is known that constitutive relations of a relativistic fluid at local thermodynamic equilibrium can be obtained from an equilibrium partition function up to some undetermined “transport coefficients” [21,22]. These coefficients can be determined either from experiments or through a microscopic calculation. If we think of Galilean fluid as a limit of an underlying relativistic theory, we would expect that its constitutive relations will also follow from such an equilibrium partition function, which has been discussed in [6]. We expect that a similar partition function can also be achieved via light cone reduction by setting the theory on background Eq. (2) to be independent of t direction. In this configuration symmetries of the theory break down to $\text{diff} \times U(1)^3$ (spatial diffeomorphisms, Kaluza-Klein transformations, mass transformation and gauge transformation), and one can easily write down the equilibrium partition function invariant under these symmetries as a gauge invariant scalar made out of the background fields. In null background picture, the same story follows by introducing another isometry K^M , and choosing a basis such that $K = \partial_t$.

Hence the refined goal of this paper is to *set up a consistent theory of hydrodynamics on null backgrounds. We want to find the most generic constitutive relations for a fluid on null backgrounds constrained by the second law of thermodynamics and requirement of an equilibrium partition function. Later employing light cone reduction, we interpret these null fluid constitutive relations as constitutive relations of the most generic Galilean fluid.*

Before closing the introduction, we would like to outline a comparison with some previous works on similar lines. We mentioned in the beginning that [2] constructed a Milne boost invariant formalism of Galilean hydrodynamics using an “extended space representation.” On taking a closer look we realize that their construction is just a “bottom-top” viewpoint of our “null fluid” where authors started with a Galilean fluid and worked out the corresponding null fluid. In this paper, we take a more axiomatic point of view and define the null fluid in its own right and realize it as a Milne

boost invariant representation of the Galilean fluid. Realizing null fluids as a self-consistent theory of fluids on a relativistic background admitting a null Killing vector, makes it technically more approachable, since we have all the machinery of relativistic hydrodynamics (known from past few years) at our disposal. We just need to tweak the relativistic fluid suitably to get the null fluid of our interest, which in turn is just a different representation of a Galilean fluid in one lower dimension.

In this paper, apart from rediscovering the known results of [2,6] for Galilean fluids in a Milne boost covariant manner, we have also used the null fluid approach to write down the most generic constitutive relations of a “charged Galilean fluid” in arbitrary (odd and even) number of dimensions, up to first order in derivatives. Further, we have discussed the potential $U(1)$ anomalies in Galilean theories (without dwelling into their field theoretic interpretation).

The organization of this paper is as follows. In Sec. II we review the construction of torsionless null backgrounds, and construct an equilibrium partition function for null theories. Then in Sec. III we study hydrodynamics on these null backgrounds, and put constraints on its dynamics by equilibrium partition function and second law of thermodynamics. We devote Sec. IV to review the procedure to obtain Galilean theories from null theories by light cone reduction, and use it to study Galilean hydrodynamics in Sec. V. Finally in Sec. VI we extend this entire construction to anomalous fluids. In Appendix A we extend the entropy current calculation in presence of minimal compatible torsion, which is required to get agreement between equilibrium partition function and entropy current constraints. In Appendix B we express all these results in conventional noncovariant basis. In Appendix [2] we provide a comparison of our results with those of [2]. At the end, in Appendix D we mention notations and conventions of differential forms used throughout this paper.

II. CONSTRUCTION OF NULL BACKGROUNDS

We start our discussion by formally setting up null backgrounds, which will prove to be a natural “embedding” of Galilean (Newton-Cartan) backgrounds into a spacetime of one higher dimension. These kind of backgrounds were first considered in [11] and further explored by [12,13,17] where authors recovered Newton-Cartan gravity by light cone reduction of general relativity. We will refine the approach by constraining the background field content so that it exactly matches that of a Galilean theory, hence letting us study physically realizable Galilean fluids later.

Let us consider a manifold $\mathcal{M}_{(d+2)}$ equipped with a metric $ds^2 = G_{MN}dx^M dx^N$ and a $U(1)$ gauge field $\mathcal{A} = \mathcal{A}_M dx^M$ together referred as *background fields/sources*. $\mathcal{M}_{(d+2)}$ is also provided with the Levi-Civita connection,

$$\Gamma^R_{MS} = \frac{1}{2} G^{RN} (\partial_M G_{NS} + \partial_S G_{NM} - \partial_N G_{MS}), \quad (5)$$

and a covariant derivative ∇_M associated with Γ^R_{MS} and \mathcal{A}_M . We demand that physical theories on $\mathcal{M}_{(d+2)}$ are left invariant by diffeomorphisms and gauge transformations parametrized by infinitesimal parameters $\psi_\xi = \{\xi = \xi^M \partial_M, \Lambda_{(\xi)}\}$ which we call *symmetry data*. Action of ψ_ξ (denoted by δ_ξ) on various background fields is given as,

$$\begin{aligned} \delta_\xi G_{MN} &= \mathcal{L}_\xi G_{MN} = 2\nabla_{(M} \xi_{N)}, \\ \delta_\xi \mathcal{A}_M &= \partial_M (\Lambda_{(\xi)} + \xi^N \mathcal{A}_N) + \xi^N \mathcal{F}_{NM}, \end{aligned} \quad (6)$$

where \mathcal{L}_ξ denotes the Lie derivative along ξ and \mathcal{F}_{MN} is the field strength of \mathcal{A}_M . One can check that symmetry data ψ_ξ form an algebra with commutator defined by,

$$\begin{aligned} \psi_{[\xi_1, \xi_2]} &\equiv [\psi_{\xi_1}, \psi_{\xi_2}] = \delta_{\xi_1} \psi_{\xi_2} - \delta_{\xi_2} \psi_{\xi_1} \\ &= \{\xi_{\xi_1} \xi_2 = -\xi_{\xi_2} \xi_1, \mathcal{L}_{\xi_1} \Lambda_{(\xi_2)} - \mathcal{L}_{\xi_2} \Lambda_{(\xi_1)}\}. \end{aligned} \quad (7)$$

Correspondingly their action on a general field (suppressing all the indices) φ also forms an algebra with commutator given by $[\delta_{\xi_1}, \delta_{\xi_2}] \varphi = \delta_{[\xi_1, \xi_2]} \varphi$. Physical theories on $\mathcal{M}_{(d+2)}$ can be described by a generating functional⁶ $W[G_{MN}, \mathcal{A}_M]$ which is seen as a functional of the background sources. Under infinitesimal variation of these sources linear response of W is captured by,

$$\delta W = \int \{dx^M\} \sqrt{-G} \left[\frac{1}{2} T^{MN} \delta G_{MN} + J^M \delta \mathcal{A}_M \right]. \quad (8)$$

T^{MN} , J^M are called energy-momentum tensor/current and charge current respectively. Demanding partition function to be invariant under the action of ψ_ξ given in Eq. (6), we can obtain a set of *Ward identities* these currents must follow,

$$\nabla_M T^{MN} = \mathcal{F}^{NM} J_M, \quad \nabla_M J^M = 0. \quad (9)$$

These are the energy-momentum and charge conservation laws of a relativistic theory. It is not mandatory for a physical theory to admit a Lagrangian description, in which case the theory itself can be characterized in terms of conserved currents T^{MN} , J^M with dynamics provided by *equations of motion* (9).

A. Compatible null isometry

So far whatever we have said applies to any relativistic theory. We now specialize to our case of interest—“null backgrounds” by introducing a null Killing vector. More formally, a symmetry data $\psi_V = \{V = V^M \partial_M, \Lambda_{(V)}\}$ will

be said to generate a *compatible null isometry* on $\mathcal{M}_{(d+2)}$ if it follows,

- (1) Action of ψ_V is an isometry, $\delta_V G_{MN} = \delta_V \mathcal{A}_M = 0$,
- (2) V is null, $V^M V_M = 0$,
- (3) V is preserved under covariant transport, $\nabla_M V^N = 0$, and,
- (4) Component of gauge field \mathcal{A} along V is fixed to⁷: $V^M \mathcal{A}_M = -\Lambda_{(V)}$.

We will call backgrounds admitting a compatible null isometry to be *null backgrounds*. Since we are working with torsionless manifolds, one can check that above conditions imply that $\mathcal{H}_{MN} = \partial_M V_N - \partial_N V_M = 0$. This is a dynamic constraint and can be violated by quantum fluctuations off-shell—a fact that will become important when we write equilibrium partition function for fluids on null backgrounds in Sec. II B 1.

Null backgrounds possess some nice features, the first one being: $V^M \nabla_M \varphi = \delta_V \varphi$ for any contra-covariant tensor φ (all indices suppressed), transforming in appropriate representation of the gauge group. Further if φ is entirely made up of G_{MN} , \mathcal{A}_M , by first consistency condition, $V^M \nabla_M \varphi = \delta_V \varphi = 0$. These consistency conditions also imply,

$$V^N \mathcal{F}_{NM} = V^N \mathcal{R}_{NMRS} = \mathcal{R}_{NMRS} V^R = 0, \quad (10)$$

where \mathcal{R}_{MNRS} is the Riemann curvature tensor. We term physical theories on null backgrounds (with compatible null isometry ψ_V) as *null theories*, and demand them to be invariant under ψ_V preserving symmetry transformations i.e. $[\psi_\xi, \psi_V] = 0$. This will break down the Poincaré symmetry algebra to Galilean, and give null theories a Galilean interpretation. Algebraic relations (2) and (4) in the definition of compatible null isometry, will imply,

$$\begin{aligned} \delta(V^M V_M) &= 0 \Rightarrow V^M V^N \delta G_{MN} = -2V_M \delta V^M, \\ \delta(V^M \mathcal{A}_M + \Lambda_{(V)}) &= 0 \Rightarrow V^M \delta \mathcal{A}_M = -\delta \Lambda_{(V)} - \mathcal{A}_M \delta V^M. \end{aligned} \quad (11)$$

It immediately follows that under a variation of background sources restricted by $\delta \psi_V = 0$, linear response of partition function Eq. (8) is still completely characterized by T^{MN} , J^M , with an added ambiguity in currents,

$$T^{MN} \rightarrow T^{MN} + \theta_1 V^M V^N, \quad J^M \rightarrow J^M + \theta_2 V^M, \quad (12)$$

where θ 's are some arbitrary scalars. One can check that they leave the conservation equations (9) invariant,

⁷Note that $V^M \mathcal{A}_M + \Lambda_{(V)}$ is a gauge invariant quantity [it can be verified using transformations Eq. (7)]; hence setting it to zero is *not a gauge fixing*. However, we will shortly choose $\Lambda_{(V)} = 0$, which is a gauge fixing, and this condition will imply $V^M \mathcal{A}_M = 0$.

⁶Actually $W = \ln \mathcal{Z}$ where \mathcal{Z} is the QFT generating functional.

provided $\delta_V \theta_1 = \delta_V \theta_2 = 0$. This point onward whenever we talk about variation, we implicitly assume it to follow $\delta \psi_V = 0$. Also for the following analysis, we partially fix ψ_V by choosing $\Lambda_{(V)} = 0$ for convenience.

B. Equilibrium

For our later discussion on equilibrium partition function of hydrodynamics, it will be helpful to define a notion of equilibrium on null backgrounds. The discussion here is hugely motivated from the equilibrium partition function results for relativistic hydrodynamics in [21,22]. A system is said to be in equilibrium if it admits a timelike isometry generated by $\psi_K = \{K = K^M \partial_M, \Lambda_{(K)}\}$, i.e. $K^M K_M < 0$ and $\delta_K G_{MN} = \delta_K \mathcal{A}_M = 0$. Using K we can define a null field,

$$\bar{V}_{(K)}^M = e^\Phi (K^M + \mathcal{B}_t V^M), \quad e^\Phi = -\frac{1}{K^M V_M},$$

$$\mathcal{B}_t = -\frac{K_M K^M}{2K_N V^N}, \quad (13)$$

which is orthonormal to V , i.e. $\bar{V}_{(K)}^M V_M = -1$, $\bar{V}_{(K)}^M \bar{V}_{(K)M} = 0$. We define *spatial slice* $\mathcal{M}_{(d)}$ as the

spacetime transverse to V and $\bar{V}_{(K)}$ with projection operator,

$$P_{(K)}^{MN} = G^{MN} - 2V^{(M} \bar{V}_{(K)}^{N)}. \quad (14)$$

Using diffeomorphism and gauge invariance of $\mathcal{M}_{(d+2)}$ we pick up coordinates $x^M = \{x^-, t, x^i\}$ such that,

$$\psi_V = \{V = \partial_-, \Lambda_{(V)} = 0\}, \quad \psi_K = \{K = \partial_t, \Lambda_{(K)} = 0\}, \quad (15)$$

and coordinates $\vec{x} = \{x^i\}$ span $\mathcal{M}_{(d)}^K$. In this basis, background fields can be decomposed as,

$$ds_{(d+2)}^2 = G_{MN} dx^M dx^N = -2e^{-\Phi} (dt + a_i dx^i) \times (dx^- - \mathcal{B}_t dt - \mathcal{B}_i dx^i) + g_{ij} dx^i dx^j,$$

$$\mathcal{A} = \mathcal{A}_M dx^M = \mathcal{A}_t dt + \mathcal{A}_i dx^i. \quad (16)$$

Indices on $\mathcal{M}_{(d)}^K$ can be raised and lowered by g^{ij} and its inverse g_{ij} . Decomposition of other derived fields follow trivially from here,

$$V_M = \begin{pmatrix} 0 \\ -e^{-\Phi} \\ -e^{-\Phi} a_i \end{pmatrix}, \quad \bar{V}_{(K)}^M = \begin{pmatrix} e^\Phi \mathcal{B}_t \\ e^\Phi \\ 0 \end{pmatrix}, \quad \bar{V}_{(K)M} = \begin{pmatrix} -1 \\ \mathcal{B}_t \\ \mathcal{B}_i \end{pmatrix},$$

$$P_{(K)MN} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g_{ij} \end{pmatrix}, \quad P_{(K)}^{MN} = \begin{pmatrix} B_k B^k & -a_k B^k & B^j \\ -a_k B^k & a^k a_k & -a^j \\ B^i & -a^i & g^{ij} \end{pmatrix}, \quad (17)$$

where we have defined $B_i = \mathcal{B}_i - a_i \mathcal{B}_t$, $A_i = \mathcal{A}_i - a_i \mathcal{A}_t$. Under this choice of basis, one can check that residual symmetry transformations are parametrized by \vec{x} dependent symmetry data $\psi_\xi = \{\xi^t, \xi^-, \vec{\xi}, \Lambda_{(\xi)}\}$, which acts on reduced set of background fields as,

$$\delta_\xi \Phi = \mathfrak{L}_{\vec{\xi}} \Phi, \quad \delta_\xi a_i = \partial_i \xi^t + \mathfrak{L}_{\vec{\xi}} a_i, \quad \delta_\xi \mathcal{B}_t = \mathfrak{L}_{\vec{\xi}} \mathcal{B}_t, \quad \delta_\xi \mathcal{B}_i = -\partial_i \xi^- + \mathfrak{L}_{\vec{\xi}} \mathcal{B}_i, \quad \delta_\xi g_{ij} = \mathfrak{L}_{\vec{\xi}} g_{ij},$$

$$\delta_\xi \mathcal{A}_t = \mathfrak{L}_{\vec{\xi}} \mathcal{A}_t, \quad \delta_\xi A_i = \mathfrak{L}_{\vec{\xi}} A_i + \partial_i \Lambda_{(\xi)},$$

where $\mathfrak{L}_{\vec{\xi}}$ denotes lie derivative with respect to $\vec{\xi}$. Its trivial to see that a_i, B_i, A_i transform as $U(1)$ vector gauge fields, $\Phi, \mathcal{B}_t, \mathcal{A}_t$ transform as scalars, and g_{ij} transform as rank 2 tensor. The response of partition function Eq. (8) in equilibrium under infinitesimal variation of these sources can be worked out to be,

$$\delta W^{eqb} = \int \{dx^i\} \sqrt{g} \left[e^\Phi (T_{t-} + T_{--} \mathcal{B}_t) \frac{1}{\vartheta_o^2} \delta \vartheta_o + \frac{1}{\vartheta_o} [T^i_t + J^i \mathcal{A}_t] \delta a_i \right. \\ \left. + \frac{1}{2\vartheta_o} T^{ij} \delta g_{ij} + \left(T_{--} \delta \varpi_o - \frac{1}{\vartheta_o} T^i_- \delta \mathcal{B}_i \right) - \left(J_- \delta \nu_o - \frac{1}{\vartheta_o} J^i \delta A_i \right) \right], \quad (18)$$

where we have defined:

$$\vartheta_o = \tilde{\vartheta} e^\Phi, \quad \varpi_o = \frac{1}{\tilde{\vartheta}} \mathcal{B}_t, \quad \nu_o = \frac{1}{\tilde{\vartheta}} \mathcal{A}_t. \quad (19)$$

$\tilde{\vartheta} = 1/(\tilde{\beta} \tilde{R})$ where $\tilde{\beta}$ is the radius of the Euclidean time $\tau = it$ and \tilde{R} is the radius of compactified x^- . We define a connection on $\mathcal{M}_{(d)}$ as,

$$\gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_k g_{ij} - \partial_l g_{ij}), \quad (20)$$

and ∇_i as its associated covariant derivative. We call the associated Riemann curvature tensor R_{ijkl} . Note that condition $\mathcal{H}_{MN} = 0$ implies that in equilibrium,

$$f_{ij} = \partial_i a_j - \partial_j a_i = 0, \quad \partial_i \vartheta_o = 0. \quad (21)$$

Again, these conditions can be violated off-shell, which will be important in the next subsection when we start construction on the equilibrium partition functions.

1. Constructing equilibrium partition function

Motivated by applications in hydrodynamics, we want to write the most generic form of the equilibrium partition function allowed by symmetries arranged in a derivative

expansion of the background sources. Look at [21,22] for a similar discussion on relativistic hydrodynamics. The partition function is generally written as integration of scalar densities. While the partition function is itself invariant under symmetries, such a statement cannot be made for the integrand. In fact, terms can be added to it whose variation is gauge invariant only up to some boundary terms. We can hence decompose W^{eqb} into,⁸

$$W^{eqb} = W^{eqb}_{H_S} + W^{eqb}_{H_V}, \quad W^{eqb}_{H_S} = \int \{dx^i\} \sqrt{g} \frac{1}{\vartheta_o} P_{H_S},$$

$$W^{eqb}_{H_V} = - \int_{\mathcal{M}_{(d)}} \mathbf{I}_{CS}^{(d)}, \quad (22)$$

where P_{H_S} is a gauge invariant scalar. $\mathbf{I}_{CS}^{(d)}$ on the other hand is the d -dimensional gauge noninvariant “Chern-Simons” form which is written such that its gauge variation is purely a boundary term. It is known that $\mathbf{I}_{CS}^{(2n-1)}$ only exists in odd number of dimensions, and for $d = 2n - 1$ up to first nontrivial order in derivatives is given as,

$$\mathbf{I}_{CS}^{(2n-1)} = \sum_{r=1}^n \binom{n}{r} \sum_{s=0}^{n-r} \binom{n-r}{s} C_{(r,s)} \mathbf{A} \wedge (\mathbf{dA})^{\wedge(r-1)} \wedge (\mathbf{dB})^{\wedge s} \wedge (\tilde{\vartheta} \mathbf{dA})^{\wedge(n-r-s)}$$

$$+ \sum_{s=1}^n \binom{n}{s} C_{(0,s)} \mathbf{B} \wedge (\mathbf{dB})^{\wedge(s-1)} \wedge (\tilde{\vartheta} \mathbf{dA})^{\wedge(n-s)} + C_{(0,0)} \tilde{\vartheta} \mathbf{A} \wedge (\tilde{\vartheta} \mathbf{dA})^{\wedge(n-1)}, \quad (23)$$

where C ’s are constants and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the binomial coefficient introduced for later convenience. Note that in writing $\mathbf{I}_{CS}^{(2n-1)}$ we have left out terms which can be related to the existing ones up to a total derivative. At higher order in derivatives there can be terms involving the curvature and affine connection as well. Note also that the condition $\mathcal{H} = 0$ would imply $\mathbf{f} = \mathbf{dA} = 0$, but since partition functions are to be written off-shell, we include these terms. From here we can find the variation of $W^{eqb}_{H_V}$ imposing $\mathbf{f} = 0$ and ignoring some boundary terms,

$$\delta W^{eqb}_{H_V} = - \int_{\mathcal{M}_{(d)}} n \sum_{r=0}^{n-1} \binom{n-1}{r} (\mathbf{dA})^{\wedge r} \wedge (\mathbf{dB})^{\wedge(n-r-1)} \wedge (C_{1,(r)} \tilde{\vartheta} \mathbf{dA} + C_{2,(r+1)} \delta \mathbf{A} + C_{2,(r)} \delta \mathbf{B}), \quad (24)$$

where $C_{1,(r)} = C_{(r,n-r-1)}$ and $C_{2,(r)} = C_{(r,n-r)}$. From here we can trivially read out the contribution of $W^{eqb}_{H_V}$ to currents in equilibrium; we will come back to it in Sec. III C 1. Coming back to $W^{eqb}_{H_S}$, it is now just integration of the most generic scalar P_{H_S} made out of background sources arranged in a derivative expansion. At ideal order (no derivatives) $P_{H_S, \text{ideal}} = P_o$ is defined as a gauge invariant function of $\vartheta_o, \varpi_o, \nu_o$,

$$W^{eqb}_{H_S, \text{ideal}} = \int \{dx^i\} \sqrt{g} \frac{1}{\vartheta_o} P_o(\vartheta_o, \varpi_o, \nu_o). \quad (25)$$

We take this opportunity to define near equilibrium thermodynamics on null backgrounds. It is known that Euclidean partition function $\mathcal{Z} = e^W$ can be identified with grand canonical partition function of statistical mechanics. It follows that P can be identified as *pressure density*, ϑ as *temperature*, and ν, ϖ as *chemical potentials* scaled with temperature, which boil down to their $_o$ values at equilibrium. Differential of $P(\vartheta, \varpi, \nu)$ can be expanded as,

$$dP = Sd\vartheta + Rd(\varpi\vartheta) + Qd(\nu\vartheta), \quad (26)$$

⁸Usage of subscripts H_S, H_V is motivated from eightfold classification of relativistic transport in [23]. It is yet not clear if such a classification is also applicable to null backgrounds, so for us this usage is purely notational.

where we identify S as *entropy density*, R as “*mass density*” and Q as *charge density*. We can also define an *energy density* E by invoking Gibbs-Duhem relation,

$$E = S\vartheta + \vartheta R\varpi + \vartheta Q\nu - P. \quad (27)$$

Taking a derivative of this relation and using Eq. (26) we can find the first law of thermodynamics,

$$dE = \vartheta dS + \vartheta \varpi dR + \vartheta \nu dQ. \quad (28)$$

Existence of a mass density makes this thermodynamic system already look Galilean and we take it as first hint that (at least) near equilibrium theories on null backgrounds are secretly Galilean.⁹ Coming back to equilibrium, we can vary partition function Eq. (25) and use Eq. (18) to read out components of currents in equilibrium at ideal order,

$$(T^{ij})_{o,\text{ideal}} = P_o g^{ij}, \quad (T_{--})_{o,\text{ideal}} = R_o, \\ e^\Phi (T_{t-} + T_{--} \mathcal{B}_t)_{o,\text{ideal}} = E_o, \quad -(J_-)_{o,\text{ideal}} = Q_o, \quad (30)$$

and rest all spatial currents zero. Using V^M , $\bar{V}_{(K)}^M$, $P_{(K)}^{MN}$ we can recompile these into covariant language,

$$T_{o,\text{ideal}}^{MN} = R_o \bar{V}_{(K)}^M \bar{V}_{(K)}^N + 2E_o V^{(M} \bar{V}_{(K)}^{N)} + P_o P_{(K)}^{MN}, \\ J_{o,\text{ideal}}^M = Q_o \bar{V}_{(K)}^M. \quad (31)$$

These look like some sort of ideal fluid constitutive relations, but are quite different from a relativistic fluid.

⁹It is interesting to see that a null theory satisfies different thermodynamics than a relativistic theory. It was noted in [15] that if we start with a relativistic fluid (following relativistic thermodynamics), thermodynamics of Galilean fluid after reduction gets restricted. In our setting this restriction manifests itself as $E + P + R\vartheta\varpi = 0$. After a nontrivial redefinition of thermodynamic functions,

$$E_{\text{rel}} = 2E + P, \quad P_{\text{rel}} = P, \quad S_{\text{rel}} = \frac{1}{a}S, \quad Q_{\text{rel}} = \frac{1}{a}Q, \\ \vartheta_{\text{rel}} = a\vartheta, \quad \mu_{\text{rel}} = \vartheta\nu, \quad \text{where } a = \frac{1}{\sqrt{-2\vartheta\varpi}},$$

this restricted thermodynamics is equivalent to relativistic thermodynamics,

$$dE_{\text{rel}} = \vartheta_{\text{rel}} dS_{\text{rel}} + \mu_{\text{rel}} dQ_{\text{rel}}, \quad E_{\text{rel}} = S_{\text{rel}}\vartheta_{\text{rel}} + Q_{\text{rel}}\mu_{\text{rel}} - P_{\text{rel}}. \quad (29)$$

Interestingly this map between relativistic and restricted null thermodynamics is exactly the same as the map between relativistic and restricted Galilean thermodynamics found in [15] by null reduction with $a = “u^+”$ in their language. It motivates us to propose that thermodynamic systems on null backgrounds are equivalent to thermodynamic systems on Galilean backgrounds.

We will make the notion of this fluid on null backgrounds—*null fluids* more precise in next section.

III. HYDRODYNAMICS ON NULL BACKGROUNDS

Having already developed some intuition in last section, we proceed to formally construct hydrodynamics on null backgrounds in its full generality. The discussion here is hugely motivated from the relativistic hydrodynamics (see e.g. [24]). Any quantum field theory in near equilibrium regime can be described by hydrodynamics. Systems having a hydrodynamic description (called *fluids*) are assumed to be in a local thermodynamic equilibrium, i.e. the spacetime variation of the fields describing the system happens on the scales much much larger than the characteristic scales of the system. This essentially means that the spacetime derivatives of the fields describing the fluid are much much smaller than the fields themselves. One can therefore express observables (currents) of the theory as a derivative expansion of symmetry covariant data made out of fluid variables.

Note that conservation laws (9) are $(d+3)$ independent equations, so any system with $(d+3)$ variables would be exactly solvable. We choose to describe our system by a fluid with null velocity u normalized as $u^M u_M = 0$, $u^M V_M = -1$, which will give us (d) degrees of freedom, and three thermodynamic variables: temperature ϑ , mass chemical potential $\vartheta\varpi$, and charge chemical potential $\vartheta\nu$. We are interested in configurations which respect the isometry generated by ψ_V . Hence, all the fluid variables as well as constitutive relations are annihilated by action of ψ_V .

Hydrodynamics (due to dissipation) is not described by a partition function; rather it is characterized by the most generic form of currents T^{MN} , J^M in terms of background fields G_{MN} , \mathcal{A}_M , ψ_V and fluid variables u^M , ϑ , ϖ , ν known as “*constitutive relations*” of the fluid. Dynamics of these currents is given by Ward identities Eq. (9) imposed as equations of motion. The constitutive relations are further constrained by certain physicality arguments, like the second law of thermodynamics or existence of an equilibrium configuration. Using u^M , V^M and $P^{MN} = G^{MN} + 2V^{(M} u^{N)}$ we can decompose constitutive relations as,

$$T^{MN} = \mathcal{R}u^M u^N + 2\mathcal{E}u^{(M} V^{N)} + \mathcal{P}P^{MN} + 2\mathbb{R}^{(M} u^{N)} \\ + 2\mathbb{E}^{(M} V^{N)} + \mathbb{T}^{MN}, \\ J^M = \mathcal{Q}u^M + \mathbb{J}^M, \quad (32)$$

where we have used redefinitions Eq. (12) to get rid of some terms. \mathcal{R} , \mathcal{E} , \mathcal{P} , \mathcal{Q} are some arbitrary functions of ϑ , ϖ , ν . The tensors \mathbb{R}^M , \mathbb{E}^M , \mathbb{T}^{MN} , \mathbb{J}^M contains derivative corrections and are transverse to u^M and V^M , and \mathbb{T}^{MN} is traceless. Comparing the constitutive relations Eq. (32) to

Eq. (31) we can infer that, at ideal order \mathcal{R} , \mathcal{E} , \mathcal{P} , \mathcal{Q} boil down to respective thermodynamic variables R , E , P , Q . In the presence of dissipation however, these functions can deviate from their thermodynamic values.

A. Hydrodynamic frames

Note that fluid variables u^M , ϑ , ϖ , ν are some arbitrary dynamical fields introduced to describe the near equilibrium quantum system. Like any field theory, these fields can be subjected to arbitrary field redefinition, called the *hydrodynamic redefinition freedom*. Some of this freedom is already fixed by the ideal order equilibrium partition function, requiring that these fields boil down to $\bar{V}_{(K)}^M$, ϑ_o , ϖ_o , ν_o in equilibrium configuration at ideal order. Away

from equilibrium however we are free to perturb these variables the way we like as long as the mentioned restriction holds,

$$\begin{aligned} u^M &\rightarrow u'^M = u^M + \delta u^M, & \vartheta &\rightarrow \vartheta' = \vartheta + \delta\vartheta, \\ \varpi &\rightarrow \varpi' = \varpi + \delta\varpi, & \nu &\rightarrow \nu' = \nu + \delta\nu, \end{aligned} \quad (33)$$

where the variations are some arbitrary functions of fluid variables and background fields, subjected to velocity normalization conditions $u_M \delta u^M = V_M \delta u^M = 0$. Note that near equilibrium assumption requires these variations should contain at least one derivative. Under these transformations, constitutive relations Eq. (32) transform up to first order in derivatives as,

$$\begin{aligned} T'^{MN}(\vartheta, \varpi, \nu) &\rightarrow T'^{MN}(\vartheta', \varpi', \nu') = T^{MN}(\vartheta, \varpi, \nu) + \delta T^{MN}(\vartheta, \varpi, \nu) + \mathcal{O}(\partial^2), \\ J^M(\vartheta, \varpi, \nu) &\rightarrow J'^M(\vartheta', \varpi', \nu') = J^M(\vartheta, \varpi, \nu) + \delta J^M(\vartheta, \varpi, \nu) + \mathcal{O}(\partial^2). \end{aligned} \quad (34)$$

Here $\delta T^{MN}(\vartheta, \varpi, \nu)$, $\delta J^M(\vartheta, \varpi, \nu)$ contain only one derivative corrections to T^{MN} , J^M under the proposed hydrodynamic field redefinition Eq. (33). It is however interesting to note that $\delta T^{MN}(\vartheta, \varpi, \nu)$, $\delta J^M(\vartheta, \varpi, \nu)$ only get contribution from the respective ideal (zero derivative) parts $\delta T_{\text{ideal}}^{MN}(\vartheta, \varpi, \nu)$, $\delta J_{\text{ideal}}^M(\vartheta, \varpi, \nu)$, because the dissipative parts are already at least one order in derivatives, and cannot admit any further corrections up to first order in derivatives. It follows that,

$$\begin{aligned} T'^{MN} &= T^{MN} + \delta T_{\text{ideal}}^{MN} \\ &= (\mathcal{R} + \delta\mathcal{R})u^M u^N + 2(\mathcal{E} + \delta\mathcal{E})u^{(M} V^{N)} + (\mathcal{P} + \delta\mathcal{P})P^{MN} \\ &\quad + 2(\mathbb{R}^{(M} + R\delta u^{(M} u^{N)} + 2(\mathbb{E}^{(M} + (E + P)\delta u^{(M} V^{N)} + \mathbb{T}^{MN}, \\ J'^M &= (\mathcal{Q} + \delta\mathcal{Q})u^M + \mathcal{Q}\delta u^M + \mathbb{J}^M, \end{aligned} \quad (35)$$

Here $\delta\mathcal{R}$, $\delta\mathcal{E}$, $\delta\mathcal{P}$, $\delta\mathcal{Q}$ denote one derivative correction to the thermodynamic variables R , E , P , Q under the transformations Eq. (33). If following Eq. (32), we denote the primed constitutive relations as,

$$\begin{aligned} T'^{MN} &= \mathcal{R}_{\text{new}}u^M u^N + 2\mathcal{E}_{\text{new}}u^{(M} V^{N)} + \mathcal{P}_{\text{new}}P^{MN} + 2\mathbb{R}_{\text{new}}^{(M} u^{N)} + 2\mathbb{E}_{\text{new}}^{(M} V^{N)} + \mathbb{T}_{\text{new}}^{MN}, \\ J'^M &= \mathcal{Q}_{\text{new}}u^M + \mathbb{J}_{\text{new}}^M, \end{aligned} \quad (36)$$

up to first order in derivatives we can obtain the *hydrodynamic frame transformations*,

$$\begin{aligned} \mathcal{R}_{\text{new}} &= \mathcal{R} + \delta\mathcal{R}, & \mathcal{E}_{\text{new}} &= \mathcal{E} + \delta\mathcal{E}, & \mathcal{P}_{\text{new}} &= \mathcal{P} + \delta\mathcal{P}, & \mathcal{Q}_{\text{new}} &= \mathcal{Q} + \delta\mathcal{Q}, & \mathbb{T}_{\text{new}}^{MN} &= \mathbb{T}^{MN}, \\ \mathbb{J}_{\text{new}}^M &= \mathbb{J}^M + \mathcal{Q}\delta u^M, & \mathbb{R}_{\text{new}}^M &= \mathbb{R}^M + R\delta u^M, & \mathbb{E}_{\text{new}}^M &= \mathbb{E}^M + (E + P)\delta u^M. \end{aligned} \quad (37)$$

Out of these we can construct three hydrodynamic frame invariants, i.e. quantities that do not transform under hydrodynamic frame transformations,

$$\begin{aligned} \Pi^{MN} &= \mathbb{T}^{MN} + P^{MN} \left[(\mathcal{P} - P) - (\mathcal{E} - E) \frac{\partial}{\partial E} P - (\mathcal{R} - R) \frac{\partial}{\partial R} P - (\mathcal{Q} - Q) \frac{\partial}{\partial Q} P \right], \\ \Upsilon^M &= \mathbb{J}^M - \frac{\mathcal{Q}}{R} \mathbb{R}^M, & \mathcal{E}^M &= \mathbb{E}^M - \frac{E + P}{R} \mathbb{R}^M. \end{aligned} \quad (38)$$

All the physical information about fluid constitutive relations is encoded in these invariants. It is sometimes convenient to fix a *hydrodynamic frame* to be able to talk about the physical constitutive relations directly. Most popular choices involve identifying \mathcal{E} , \mathcal{R} , \mathcal{Q} with the thermodynamic variables E , R , Q , and keeping all the dissipation in \mathcal{P} . This fixes the ambiguity in ϑ , ϖ , ν .

For fixing the velocity redefinition, in spirit with the “usual” relativistic fluids we can use, “Eckart frame” in which \mathbb{J}^M is chosen to be zero, or “Landau frame” in which \mathbb{E}^M is zero. A more natural¹⁰ frame in this case is the ‘Mass Frame’ where \mathbb{R}^M is chosen to be zero, which aligns velocity along R flow and all dissipation transverse to V^M . We will mainly work in the mass frame for which constitutive relations are given as,

$$\begin{aligned} T^{MN} &= Ru^Mu^N + 2Eu^{(M}V^{N)} + PP^{MN} + 2\mathcal{E}^{(M}V^{N)} + \Pi^{MN}, \\ J^M &= Qu^M + \Upsilon^M. \end{aligned} \quad (39)$$

Another helpful frame for our work is to choose all the fluid variables to be equal to their values at equilibrium¹¹ exactly, not just at ideal order. We call this “equilibrium frame.” This has the advantage that equilibrium partition function naturally gives constitutive relations in this frame. To be precise, in equilibrium configuration setting $\{u^M, \vartheta, \varpi, \nu\} = \{\bar{V}_{(K)}^M, \vartheta_o, \varpi_o, \nu_o\}$ in the constitutive relations Eq. (32), and putting them into equilibrium partition function variation Eq. (18), we can deduce that,

$$\begin{aligned} \mathcal{R}_o &= \frac{\delta W_{(d)}^{eqb}}{\delta \varpi_o}, & \mathbb{R}_o^i &= \vartheta_o \frac{\delta W_{(d)}^{eqb}}{\delta B_i}, & \mathcal{Q}_o &= \frac{\delta W_{(d)}^{eqb}}{\delta \nu_o}, & \mathbb{J}_o^i &= \vartheta_o \frac{\delta W_{(d)}^{eqb}}{\delta A_i}, \\ \mathcal{E}_o &= \vartheta_o^2 \frac{\delta W_{(d)}^{eqb}}{\delta \vartheta_o}, & \mathbb{E}_o^i - \varpi_o \vartheta_o \mathbb{R}_o^i - \nu_o \vartheta_o \mathbb{J}_o^i &= -\vartheta_o e^\Phi \frac{\delta W_{(d)}^{eqb}}{\delta a_i}, & \mathcal{P}_o g^{ij} + \mathbb{T}_o^{ij} &= 2\vartheta_o \frac{\delta W_{(d)}^{eqb}}{\delta g_{ij}}. \end{aligned} \quad (40)$$

Switching back and forth between frames is a nontrivial task, and has to be done order by order in derivatives. We shall see in the subsequent sections that different physical aspects of our theory of interest are better understood in different frames. We have to switch between frames accordingly.

B. Entropy current

Since hydrodynamics is an effective field theory, we start by writing down all possible expressions, compatible with symmetry, that can contribute to $\mathbb{R}^M, \mathbb{E}^M, \mathbb{T}^{MN}, \mathbb{J}^M$. In addition since we are dealing with a thermodynamic system, we must ensure that the second law of thermodynamics is satisfied, i.e. there must exist an entropy current J_s^M , whose divergence is positive semidefinite,

$$\nabla_M J_s^M \geq 0. \quad (41)$$

We can construct the most generic entropy current for the fluid as,

$$J_s^M = J_{s,\text{can}}^M + \Upsilon_s^M, \quad J_{s,\text{can}}^M = \frac{1}{\vartheta} Pu^M - \frac{1}{\vartheta} T^{MN} u_N + \varpi T^{MN} V_N - \nu J^M, \quad (42)$$

which is just Su^M at ideal order. $J_{s,\text{can}}^M$ is called the canonical entropy current, and is given purely in terms of the constitutive relations. Υ_s^M on the other hand, are arbitrary derivative corrections to the entropy current. Note that Υ_s^M , unlike Υ^M , is not required to be transverse to u^M and V^M . Using first order equations of motion we can obtain,

$$\begin{aligned} u^M \partial_M E &= -(E + P)\Theta, & u^M \partial_M R &= -R\Theta, & u^M \partial_M Q &= -Q\Theta, \\ P^{MN} \left[R(\Omega_{NR} u^R - \vartheta \partial_N \varpi) - (E + P) \frac{1}{\vartheta} \partial_N \vartheta + Q(\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) \right] &= 0, \end{aligned} \quad (43)$$

where we have defined,

$$\Omega_{MN} = \partial_M u_N - \partial_N u_M, \quad \Theta = \nabla_M u^M. \quad (44)$$

Using these, divergence of the canonical entropy current can be computed to be,

¹⁰Upon reduction this will imply that mass current does not have any dissipation, i.e. we associate the fluid velocity with the flow of mass.

¹¹This frame choice however does not completely fix the hydrodynamic ambiguity. You can still shift fluid variables with terms that vanish in equilibrium. However for equilibrium partition function calculations, it is good enough.

TABLE I. Leading derivative order data for null fluid.

	Data	Value at Equilibrium
	Parity Even	
	$\Theta \equiv \nabla_M u^M$	0
*	$u^M \partial_M \vartheta, u^M \partial_M \varpi, u^M \partial_M \nu$	0, 0, 0
	$P^{MN} \partial_N \vartheta, P^{MN} \partial_N \varpi, P^{MN} \partial_N \nu$	0, $\nabla^i \varpi, \nabla^i \nu$
*	$P^{MN} (\Omega_{NR} u^R - \vartheta \partial_N \varpi)$	0
	$P^{MN} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)$	0
	$\sigma^{MN} \equiv 2P^{MR} P^{NS} \nabla_{(R} u_{S)} - \frac{2}{d} P^{MN} \Theta$	0
	Parity Odd—Odd Dimensions ($d = 2n - 1$)	
	$l_{(r)}^M _{r=0}^{n-1} \equiv \star[V \wedge u \wedge \mathcal{F}^{\wedge r} \wedge \Omega^{\wedge(n-r-1)}]^M$	$l_{o(r)}^i _{r=0}^{n-1} \equiv \star[(d\mathbf{A})^{\wedge r} \wedge (d\mathbf{B})^{\wedge(n-r-1)}]^i$
	Parity Odd—Even Dimensions ($d = 2n$)	
	$l_{(r)} _{r=0}^n \equiv \star[V \wedge u \wedge \mathcal{F}^{\wedge r} \wedge \Omega^{\wedge(n-r)}]$	$l_{o(r)} _{r=0}^n \equiv \star[(d\mathbf{A})^{\wedge r} \wedge (d\mathbf{B})^{\wedge(n-r)}]$
	$l_{(r)}^{MN} \partial_N \vartheta, l_{(r)}^{MN} \partial_N \varpi, l_{(r)}^{MN} \partial_N \nu$	0, $l_{(r)}^{ij} \partial_j \varpi, l_{(r)}^{ij} \partial_j \nu$
*	$l_{(r)}^{MN} (\Omega_{NR} u^R - \vartheta \partial_N \varpi)$	0
	$l_{(r)}^{MN} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)$	0
	$l_{(r)}^{R(M} \sigma^{N)}$	0
	where, $l_{(r)}^{MN} _{r=0}^{n-1} \equiv \star[V \wedge u \wedge \mathcal{F}^{\wedge r} \wedge \Omega^{\wedge(n-r-1)}]^{MN}$	$l_{o(r)}^{ij} _{r=0}^{n-1} \equiv \star[(d\mathbf{A})^{\wedge r} \wedge (d\mathbf{B})^{\wedge(n-r-1)}]^{ij}$

$$\begin{aligned} \vartheta \nabla_M J_{s(\text{can})}^M &= -\Pi^{MN} \nabla_M u_N - \frac{1}{\vartheta} \mathcal{E}^M \partial_M \vartheta \\ &+ \Upsilon^M (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu), \end{aligned} \quad (45)$$

which will come in handy later. Note that each term in the above expression is the product of derivatives (called composites). This heavily constrains the form of Υ_s^M . Its divergence must not contain any pure derivative terms (terms which are not composites), otherwise total entropy current cannot be ensured positive semidefinite.

In next subsection, we write the most generic constitutive relations of a null fluid up to leading derivative order¹² in parity-odd and even sectors. We further impose constraints on these constitutive relations by imposing second law of thermodynamics and requirement of an equilibrium partition function independently, and compare the results from both the approaches. Readers who are more interested in Galilean fluid results, can skip this computation and directly proceed to subsection III D where the final results for null fluid have been summarized. These results can be

directly used to read off the constitutive relations of a Galilean fluid, which has been done in Secs. IV and V.

C. Leading order hydrodynamics

In [25] we discussed in detail the procedure to count various independent data that appear in constitutive relations of usual relativistic fluids. This can be easily extended to a null fluid. However in this work we are only interested in leading order null fluid, so we can write the required data by hand without going into the technicalities of [25]. All possible scalars, vectors and symmetric traceless tensors made out of background fields and fluid variables has been enlisted in Table I; data marked with * can be eliminated by using first order equations of motion (43). In Table I, we have used the notation $l_{(r)}|_{r=0}^m$ which means that there are $m + 1$ distinct quantities $l_{(r)}$ parametrized by $r = 0, 1, \dots, m$.

Using data in Table I we can now write the most generic form of leading order constitutive relations. For parity-even sector we will get,

$$\begin{aligned} \Pi_{(1)}^{MN} &= -\eta \sigma^{MN} - P^{MN} \zeta \Theta, \\ \mathcal{E}_{(1)}^M &= P^{MN} [\lambda_{e\varpi} \partial_N \varpi + \lambda_{e\nu} \partial_N \nu + \kappa_e \partial_N \vartheta + \sigma_e (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)], \\ \Upsilon_{(1)}^M &= P^{MN} [\lambda_{q\varpi} \partial_N \varpi + \lambda_{q\nu} \partial_N \nu + \kappa_q \partial_N \vartheta + \sigma_q (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)]. \end{aligned} \quad (46)$$

In parity-odd sector however, in odd number of dimensions ($d = 2n - 1$) we will get,

¹²By leading order we mean the first derivative corrections appearing in constitutive relations. In the parity-even sector it happens at one derivative order itself. In the parity-odd sector however it depends on the number of dimensions—in odd dimensions ($d = 2n - 1$) the first correction appear at $(n - 1)$ derivative order, and in even dimensions ($d = 2n$) in appears at (n) derivative order.

$$\tilde{\Pi}_{(n-1)}^{MN} = 0, \quad \tilde{\mathcal{E}}_{(n-1)}^M = \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{\epsilon(r)} l_{(r)}^M, \quad \tilde{\Upsilon}_{(n-1)}^M = \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{q(r)} l_{(r)}^M, \quad (47)$$

and in even number of dimensions ($d = 2n$),

$$\begin{aligned} \tilde{\Pi}_{(n)}^{MN} &= -P^{MN} \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{(r)} l_{(r)} - \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\eta}_{(r)} l_{(r)}^{R(M} \sigma^{N)}{}_R, \\ \tilde{\mathcal{E}}_{(n)}^M &= \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} [\tilde{\lambda}_{\epsilon\varpi(r)} \partial_N \varpi + \tilde{\lambda}_{\epsilon\nu(r)} \partial_N \nu + \tilde{\kappa}_{\epsilon(r)} \partial_N \vartheta + \tilde{\sigma}_{\epsilon(r)} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)], \\ \tilde{\Upsilon}_{(n)}^M &= \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} [\tilde{\lambda}_{q\varpi(r)} \partial_N \varpi + \tilde{\lambda}_{q\nu(r)} \partial_N \nu + \tilde{\kappa}_{q(r)} \partial_N \vartheta + \tilde{\sigma}_{q(r)} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)]. \end{aligned} \quad (48)$$

Similarly we can work out constitutive relations to arbitrary high derivative orders, but in this work we will not be interested in those.

1. Constraints through equilibrium partition function

The constitutive relations as described above are constrained by the requirement of existence of an equilibrium partition function. The statement is that at equilibrium, any null theory must be determined by the most generic partition function made out of background fields discussed in Sec. II B [21,22]. We have already seen that at ideal order, the equilibrium partition function gives thermodynamic meaning to various functions. Even at further order in derivatives, the equilibrium partition function turns out to be very useful to (partially) determine the constitutive relations. It gives constraints on various transport coefficients, and tells us which of them are physical. We along with many people in past have used this approach to find transport of a relativistic fluid. Here we attempt to outline a similar procedure for null fluid up to leading order in derivatives.

Leading order parity even sector.—Leading order parity even sector contains one derivative corrections to ideal fluid dynamics. Using Table I, we see that at equilibrium only terms coupling to λ 's survive in frame invariants Eq. (46),

$$\begin{aligned} \Pi_{o(1)}^{ij} &= 0, \quad \mathcal{E}_{o(1)}^i = \lambda_{o\epsilon\varpi} \nabla^i \varpi_o + \lambda_{o\epsilon\nu} \nabla^i \nu_o, \\ \Upsilon_{o(1)}^i &= \lambda_{oq\varpi} \nabla^i \varpi_o + \lambda_{oq\nu} \nabla^i \nu_o. \end{aligned} \quad (49)$$

On the other hand there are no one-derivative scalars at equilibrium to construct partition function. Hence all the coefficients appearing above must vanish,

$$\boxed{\lambda_{\epsilon\varpi} = \lambda_{\epsilon\nu} = \lambda_{q\varpi} = \lambda_{q\nu} = 0.} \quad (50)$$

Since equilibrium partition function is identically zero, none of the fluid variables get order one even correction out of equilibrium (in mass frame).

Leading order parity odd sector (for $d = 2n - 1$).—In odd dimensions, $d = 2n - 1$, the first parity odd contributions show up at $(n - 1)$ -derivative order. At equilibrium all the parity-odd terms survive in constitutive relations Eq. (47). On the other hand there are no gauge invariant scalars to construct equilibrium partition function, and it gets contributions only from the Chern-Simons piece (cf. Sec. II B 1). Consequently we get constitutive relations in equilibrium frame,

$$\begin{aligned} \tilde{\mathbb{E}}_{o(n-1)}^i &= \vartheta_o^2 n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^i \\ &\quad \times (C_{1,(r)} - \varpi_o C_{2,(r)} - \nu_o C_{2,(r+1)}), \\ \tilde{\mathbb{R}}_{o(n-1)}^i &= -\vartheta_o n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^i C_{2,(r)}, \\ \tilde{\mathbb{J}}_{o(n-1)}^i &= -\vartheta_o n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^i C_{2,(r+1)}, \end{aligned} \quad (51)$$

and the rest are all zero. Here $C_{1,(r)}$, $C_{2,(r)}$ are constants introduced in Sec. II B 1. Performing a hydrodynamic frame transformation, we can get the transport coefficients introduced in frame invariants Eq. (47) as,

$$\begin{aligned} \tilde{\omega}_{\epsilon(r)} &= \vartheta n \left(\vartheta C_{1,(r)} + \frac{E + P - \vartheta \varpi R}{R} C_{2,(r)} - \vartheta \nu C_{2,(r+1)} \right), \\ \tilde{\omega}_{q(r)} &= \vartheta n \left(\frac{Q}{R} C_{2,(r)} - C_{2,(r+1)} \right). \end{aligned} \quad (52)$$

We see that both (set of) transport coefficients are completely determined up to some constants. Outside equilibrium, fluid velocity gets a correction (in mass frame) given by:

$$\tilde{\Delta}^{(n-1)} u^i = -\frac{\vartheta_o n}{R_o} \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^i C_{2,(r)}. \quad (53)$$

Here $\tilde{\Delta}^{(n-1)}$ represents the $(n-1)$ derivative order parity-odd correction to fluid variables in mass frame outside equilibrium. Corrections to other components of velocity can be determined by this using normalization conditions. Other fluid variables do not get any leading order odd correction.

*Leading order parity odd sector (for $d = 2n$).—*Contrary to the last case studied, in even dimensions $d = 2n$, the first parity odd contributions show up at n -derivative order. In

even number of dimensions, only terms coupling to $\tilde{\lambda}$'s and $\tilde{\zeta}$ survive in frame invariants Eq. (48),

$$\begin{aligned} \tilde{\Pi}_{o(n)}^{ij} &= -g^{ij} \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{o(r)} l_{o(r)}, \\ \tilde{\mathcal{E}}_{o(n)}^i &= \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^{ij} [\tilde{\lambda}_{oe\varpi(r)} \partial_j \varpi + \tilde{\lambda}_{oe\nu(r)} \partial_j \nu], \\ \tilde{\Upsilon}_{o(n)}^i &= \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^{ij} [\tilde{\lambda}_{oq\varpi(r)} \partial_j \varpi + \tilde{\lambda}_{oqv(r)} \partial_j \nu]. \end{aligned} \quad (54)$$

On the other hand using data in Table I, we can write the equilibrium partition function as,

$$W^{eqb} = - \int_{\mathcal{M}(d)} \sum_{r=0}^n \binom{n}{r} \sum_{s=0}^{n-r} \binom{n-r}{s} \mathcal{S}_{o(r,s)} (\mathbf{dA})^{\wedge r} \wedge (\mathbf{dB})^{\wedge s} \wedge (\tilde{\vartheta} \mathbf{da})^{\wedge (n-r-s)}. \quad (55)$$

Varying this partition function, we can compute the constitutive relations in equilibrium frame,

$$\begin{aligned} \mathcal{P}_o &= 0, \quad \mathbb{T}_o^{ij} = 0, \\ \mathcal{E}_o &= -\vartheta_o^2 \sum_{r=0}^n \binom{n}{r} \frac{\partial}{\partial \vartheta_o} \mathcal{S}_{o2,(r)} l_{o(r)}, \quad \mathcal{R}_o = -\sum_{r=0}^n \binom{n}{r} \frac{\partial}{\partial \varpi_o} \mathcal{S}_{o2,(r)} l_{o(r)}, \quad \mathcal{Q}_o = -\sum_{r=0}^n \binom{n}{r} \frac{\partial}{\partial \nu_o} \mathcal{S}_{o2,(r)} l_{o(r)}, \\ \mathbb{E}_o^i &= \vartheta_o^2 n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^{ij} (\partial_j \mathcal{S}_{o1,(r)} - \varpi_o \partial_j \mathcal{S}_{o2,(r)} - \nu_o \partial_j \mathcal{S}_{o2,(r+1)}) \\ \mathbb{R}_o^i &= -\vartheta_o n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^{ij} \partial_j \mathcal{S}_{o2,(r)} \quad \mathbb{J}_o^i = -\vartheta_o n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^{ij} \partial_j \mathcal{S}_{o2,(r+1)}, \end{aligned} \quad (56)$$

where $\mathcal{S}_{1,(r)} = \mathcal{S}_{(r,n-r-2)}$, $\mathcal{S}_{2,(r)} = \mathcal{S}_{(r,n-r-1)}$. Transforming these to mass frame, one can compute the transport coefficients appearing in Eq. (54),

$$\begin{aligned} \tilde{\zeta}_{(r)} &= - \left[\vartheta^2 \frac{\partial P}{\partial E} \frac{\partial}{\partial \vartheta} + \frac{\partial P}{\partial R} \frac{\partial}{\partial \varpi} + \frac{\partial P}{\partial Q} \frac{\partial}{\partial \nu} \right] \mathcal{S}_{2,(r)}, \\ \tilde{\lambda}_{e\varpi(r)} &= \vartheta n \left[\vartheta \frac{\partial}{\partial \varpi} \mathcal{S}_{1,(r)} + \frac{E+P-\vartheta \varpi R}{R} \frac{\partial}{\partial \varpi} \mathcal{S}_{2,(r)} - \vartheta \nu \frac{\partial}{\partial \varpi} \mathcal{S}_{2,(r+1)} \right], \\ \tilde{\lambda}_{e\nu(r)} &= \vartheta n \left[\vartheta \frac{\partial}{\partial \nu} \mathcal{S}_{1,(r)} + \frac{E+P-\vartheta \varpi R}{R} \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r)} - \vartheta \nu \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r+1)} \right], \\ \tilde{\lambda}_{q\varpi(r)} &= \vartheta n \left[\frac{Q}{R} \frac{\partial}{\partial \varpi} \mathcal{S}_{2,(r)} - \frac{\partial}{\partial \varpi} \mathcal{S}_{2,(r+1)} \right], \\ \tilde{\lambda}_{qv(r)} &= \vartheta n \left[\frac{Q}{R} \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r)} - \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r+1)} \right]. \end{aligned} \quad (57)$$

We see that 5 (set of) transport coefficients $\tilde{\zeta}_{(r)}$, $\tilde{\lambda}_{e\varpi(r)}$, $\tilde{\lambda}_{e\nu(r)}$, $\tilde{\lambda}_{q\varpi(r)}$, $\tilde{\lambda}_{qv(r)}$ are determined in terms of 2 (set of) functions $\mathcal{S}_{1,(r)}$, $\mathcal{S}_{2,(r)}$. Corrections to fluid variables outside equilibrium in mass frame are given as,

$$\begin{aligned} \tilde{\Delta}^{(n)} \vartheta &= -\sum_{r=0}^n \binom{n}{r} l_{o(r)} \left[\vartheta_o^2 \frac{\partial \vartheta_o}{\partial E_o} \frac{\partial}{\partial \vartheta_o} + \frac{\partial \vartheta_o}{\partial R_o} \frac{\partial}{\partial \varpi_o} + \frac{\partial \vartheta_o}{\partial Q_o} \frac{\partial}{\partial \nu_o} \right] \mathcal{S}_{o2,(r)}, \\ \tilde{\Delta}^{(n)} \varpi &= -\sum_{r=0}^n \binom{n}{r} l_{o(r)} \left[\vartheta_o^2 \frac{\partial \varpi_o}{\partial E_o} \frac{\partial}{\partial \vartheta_o} + \frac{\partial \varpi_o}{\partial R_o} \frac{\partial}{\partial \varpi_o} + \frac{\partial \varpi_o}{\partial Q_o} \frac{\partial}{\partial \nu_o} \right] \mathcal{S}_{o2,(r)}, \\ \tilde{\Delta}^{(n)} \nu &= -\sum_{r=0}^n \binom{n}{r} l_{o(r)} \left[\vartheta_o^2 \frac{\partial \nu_o}{\partial E_o} \frac{\partial}{\partial \vartheta_o} + \frac{\partial \nu_o}{\partial R_o} \frac{\partial}{\partial \varpi_o} + \frac{\partial \nu_o}{\partial Q_o} \frac{\partial}{\partial \nu_o} \right] \mathcal{S}_{o2,(r)}, \\ \tilde{\Delta}^{(n)} u^i &= -\frac{\vartheta_o}{R_o} n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^{ij} \partial_j \mathcal{S}_{o2,(r)}. \end{aligned} \quad (58)$$

Here $\tilde{\Delta}^{(n)}$ represents the n derivative order parity-odd correction to fluid variables in the mass frame outside equilibrium.

2. Constraints through entropy current

As we have said, the second law of thermodynamics for null fluid implies the existence of an entropy current with non-negative divergence. From our experience of usual relativistic fluids, we expect the second law requirement to give all the constraints we found through equilibrium partition function, and more [21,22]. However as we shall see, we will not get all the partition function constraints through entropy current. This can be accounted to the fact that in this computation we will miss constraints coupling

to $\mathcal{H} = dV$, which is set to zero by requirement of manifold being torsionless. Since this condition can be violated off-shell, equilibrium partition function can however “see” these constraints. In Appendix A we will turn on a minimal amount of torsion to allow nonzero \mathcal{H} , and will verify that we get all the partition function constraints through entropy current analysis as well. Here we perform torsionless computation to leading derivative order.

Leading order parity even sector.—At leading even derivative order, no terms can be introduced in Υ_s^M without having pure derivative terms in the divergence, hence, $J_s^M = J_{s(\text{can})}^M$ whose divergence using Eqs. (45), (46) is given as,

$$\begin{aligned} \vartheta \nabla_M J_s^M &= -\frac{1}{\vartheta} P^{MN} (\lambda_{e\varpi} \partial_N \varpi + \lambda_{e\nu} \partial_N \nu) \partial_M \vartheta + P^{MN} (\lambda_{e\varpi} \partial_N \varpi + \lambda_{e\nu} \partial_N \nu) (\mathcal{F}_{MR} u^R - \vartheta \partial_M \nu) \\ &\quad - \frac{1}{\vartheta} (\sigma_e - \vartheta \kappa_q) (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) P^{NM} \partial_M \vartheta \\ &\quad - \frac{1}{\vartheta} \kappa_e P^{MN} \partial_M \vartheta \partial_N \vartheta + \sigma_q P^{MN} (\mathcal{F}_{MR} u^R - \vartheta \partial_M \nu) (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) + \frac{1}{2} \eta \sigma^{MN} \sigma_{MN} + \zeta \Theta^2. \end{aligned} \quad (59)$$

Demanding $\nabla_M J_s^M \geq 0$, from the first line we get all the equilibrium partition function constraints Eq. (50), and in addition from the last two lines,

$$\boxed{\eta, \zeta, \sigma_q \geq 0, \quad \kappa_e \leq 0, \quad \sigma_e = \vartheta \kappa_q.} \quad (60)$$

Leading order parity odd sector (for $d = 2n - 1$).—In the parity odd sector however, there are terms we can write in Υ_s^M at leading derivative order which have composite divergence. We first consider the odd dimensional case for which we will get,¹³

$$\Upsilon_s^M = \sum_{r=0}^{n-1} \left[n \binom{n-1}{r} \tilde{\omega}_{s(r)} l_{(r)}^M - \binom{n}{r+1} C_{2,(r+1)} \star [\hat{\mathcal{A}} \wedge u \wedge \hat{\mathcal{F}}^{\wedge r} \wedge \hat{\Omega}^{\wedge(n-r-1)}]_M \right]. \quad (61)$$

Here $\hat{\mathcal{A}}_M = \mathcal{A}_M + \vartheta \nu V_M$, $\hat{u}_M = u_M + \vartheta \varpi V_M$ and $\hat{\mathcal{F}}_{MN}$, $\hat{\Omega}_{MN}$ are the respective field strengths. $C_{2(r)}$ are constants. One can check that no other terms are allowed. Note that since the entropy current is not a direct observable, only its divergence is, we are allowed to include gauge noninvariant terms in Υ_s^M as long as the divergence is gauge invariant. Computing the divergence of Eq. (61) we can obtain,

$$\begin{aligned} -\nabla_M \Upsilon_s^M &= l_{(0)}^M n [-\partial_M \tilde{\omega}_{s(0)} + C_{2,(1)} \nu \partial_M \vartheta - C_{2,(1)} (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu)] \\ &\quad + \sum_{r=1}^{n-1} \binom{n-1}{r} l_{(r)}^M n \left[-\nabla_M \tilde{\omega}_{s(r)} - \left(\frac{E + P - \vartheta \varpi R}{R} C_{2,(r)} - \vartheta \nu C_{2,(r+1)} \right) \frac{1}{\vartheta} \partial_M \vartheta \right. \\ &\quad \left. + \left(\frac{Q}{R} C_{2,(r)} - C_{2,(r+1)} \right) (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu) \right]. \end{aligned} \quad (62)$$

On the other hand divergence of the canonical entropy current through Eqs. (45) and (47) is given as,

¹³The C 's in this expression are arbitrary, and *a priori* have no connection to the C 's introduced in previous sections. However, as is suggested by the notation, both will eventually turn out to be the same quantities in constitutive relations. Also, the entropy current need not be gauge invariant.

$$\begin{aligned} \vartheta \nabla_M J_{s(\text{can})}^M &= \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^M \\ &\times \left[-\tilde{\omega}_{\epsilon(r)} \frac{1}{\vartheta} \partial_M \vartheta + \tilde{\omega}_{q(r)} (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu) \right]. \end{aligned} \quad (63)$$

Combining the two pieces and demanding $\nabla_M J_s^M \geq 0$, we find a consistency condition in entropy current that $\tilde{\omega}_{s(r)} = \tilde{\omega}_{s(r)}(\vartheta)$ must not be a function of ϖ, ν . From here parity odd transport coefficients in Eq. (46) are determined to be,

$$\tilde{\omega}_{\epsilon(0)} = \vartheta n (\vartheta C_{1,(0)} - \vartheta \nu C_{2,(1)}), \quad \tilde{\omega}_{q(0)} = -\vartheta n C_{2,(1)}, \quad (64)$$

and for $r \neq 0$,

$$\begin{aligned} \tilde{\omega}_{\epsilon(r)} &= \vartheta n \left(\vartheta C_{1,(r)} + \frac{E + P - \vartheta \varpi R}{R} C_{2,(r)} - \vartheta \nu C_{2,(r+1)} \right), \\ \tilde{\omega}_{q(r)} &= \vartheta n \left(\frac{Q}{R} C_{2,(r)} - C_{2,(r+1)} \right), \end{aligned} \quad (65)$$

where $C_{1,(r)} = \frac{d}{d\vartheta} \tilde{\omega}_{s(r)}(\vartheta)$ is an arbitrary function of ϑ . Compared to equilibrium partition function constraints Eq. (52), we have one additional constraint,

$$\boxed{C_{2,(0)} = 0}, \quad (66)$$

and one less constraint: $C_{1,(r)}$ is not a constant but a function of ϑ . In Appendix A we show that on introducing torsion, entropy current positivity will indeed set $C_{1,(r)}$ to be a constant.

*Leading order parity odd sector (for $d = 2n$).—*Now we perform a similar analysis for the even dimensional parity odd sector. Similar to the odd dimensional case, here also we can have terms in Υ_s^M whose divergence does not have any pure derivative terms,¹⁴

$$\begin{aligned} \Upsilon_s^M &= \sum_{r=0}^n \binom{n}{r} \mathcal{S}_{2,(r)} \star [u \wedge \hat{\mathcal{F}}^{\wedge r} \wedge \hat{\Omega}^{\wedge (n-r)}]^M \\ &+ \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \\ &\times [\tilde{\lambda}_{s\varpi(r)} \partial_N \varpi + \tilde{\lambda}_{s\nu(r)} \partial_N \nu + \tilde{\kappa}_{s(r)} \partial_N \vartheta]. \end{aligned} \quad (67)$$

One can check that any other term, if included, will give pure derivative terms in the divergence. Divergence of this object can be computed fairly easily to be,

$$\begin{aligned} \nabla_M \Upsilon_s^M &= \Theta \sum_{r=0}^n \binom{n}{r} l_{(r)} \left[\vartheta \frac{\partial P}{\partial E} \frac{\partial}{\partial \vartheta} + \frac{1}{\vartheta} \frac{\partial P}{\partial R} \frac{\partial}{\partial \varpi} + \frac{1}{\vartheta} \frac{\partial P}{\partial Q} \frac{\partial}{\partial \nu} \right] \mathcal{S}_{2,(r)} \\ &+ \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left[\left(\frac{\partial}{\partial \vartheta} \tilde{\lambda}_{s\varpi(r)} - \frac{\partial}{\partial \varpi} \tilde{\kappa}_{s(r)} + n \frac{E + P - \vartheta \varpi R}{\vartheta R} \frac{\partial}{\partial \varpi} \mathcal{S}_{2,(r)} - n \nu \frac{\partial}{\partial \varpi} \mathcal{S}_{2,(r+1)} \right) \partial_M \vartheta \partial_N \varpi \right. \\ &+ \left(\frac{\partial}{\partial \vartheta} \tilde{\lambda}_{s\nu(r)} - \frac{\partial}{\partial \nu} \tilde{\kappa}_{s(r)} + n \frac{E + P - \vartheta \varpi R}{\vartheta R} \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r)} - n \nu \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r+1)} \right) \partial_M \vartheta \partial_N \nu \\ &\left. + \left(\frac{\partial}{\partial \nu} \tilde{\lambda}_{s\varpi(r)} - \frac{\partial}{\partial \varpi} \tilde{\lambda}_{s\nu(r)} \right) \partial_M \nu \partial_N \varpi + n \left(\frac{Q}{R} \partial_M \mathcal{S}_{2,(r)} - \partial_M \mathcal{S}_{2,(r+1)} \right) (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) \right]. \end{aligned} \quad (68)$$

On the other hand, divergence of the canonical entropy current can be computed using Eqs. (45) and (48), and is given by,

$$\begin{aligned} -\vartheta \nabla_M J_{s(\text{can})}^M &= -\Theta \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{(r)} l_{(r)} + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left[\tilde{\lambda}_{\epsilon\varpi(r)} \frac{1}{\vartheta} \partial_M \vartheta \partial_N \varpi + \tilde{\lambda}_{\epsilon\nu(r)} \frac{1}{\vartheta} \partial_M \vartheta \partial_N \nu \right] \\ &+ \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left[\tilde{\lambda}_{q\varpi(r)} \partial_M \varpi + \tilde{\lambda}_{q\nu(r)} \partial_M \nu + (\vartheta \tilde{\kappa}_{q(r)} + \tilde{\sigma}_{\epsilon(r)}) \frac{1}{\vartheta} \partial_M \vartheta \right] (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu). \end{aligned} \quad (69)$$

Combining the two pieces and demanding $\nabla_M J_s^M \geq 0$, we get a consistency condition on the entropy current Eq. (67),

¹⁴The \mathcal{S} 's in this expression are arbitrary, and *a priori* have no connection to the \mathcal{S} 's introduced in previous sections. However, as is suggested by the notation, both will eventually turn out to be the same quantities in constitutive relations.

$$\frac{\partial}{\partial \nu} \tilde{\lambda}_{s\varpi(r)} = \frac{\partial}{\partial \varpi} \tilde{\lambda}_{s\nu(r)}, \quad (70)$$

whose most generic solution is,

$$\begin{aligned} \lambda_{s\varpi(r,s)} &= \frac{\partial}{\partial \varpi} f_1(\vartheta, \varpi, \nu), \\ \lambda_{s\nu(r,s)} &= \frac{\partial}{\partial \nu} f_1(\vartheta, \varpi, \nu) + \frac{\partial}{\partial \nu} f_2(\vartheta, \nu), \end{aligned} \quad (71)$$

for some functions $f_1(\vartheta, \varpi, \nu)$, $f_2(\vartheta, \nu)$. We define,

$$n\mathcal{S}_{1,(r)} = -\tilde{\kappa}_{s(r)} + \frac{\partial}{\partial \vartheta} f_1. \quad (72)$$

Expressed in these variables, one can check that the entropy current positivity gives all the partition function constraints Eq. (57), except the expression for $\tilde{\lambda}_{e\nu(r)}$ modifies to,

$$\begin{aligned} \tilde{\lambda}_{e\nu(r)} &= \vartheta n \left[\vartheta \frac{\partial^2}{\partial \nu \partial \vartheta} f_2(\vartheta, \nu) + \vartheta \frac{\partial}{\partial \nu} \mathcal{S}_{1,(r)} \right. \\ &\quad \left. + \frac{E + P - \vartheta \varpi R}{R} \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r)} - \vartheta \nu \frac{\partial}{\partial \nu} \mathcal{S}_{2,(r+1)} \right], \end{aligned} \quad (73)$$

and in addition we get,

$$\tilde{\kappa}_{q(r)} + \frac{1}{\vartheta} \tilde{\sigma}_{e(r)} = \vartheta n \left[\frac{Q}{R} \frac{\partial}{\partial \vartheta} \mathcal{S}_{2,(r)} - \frac{\partial}{\partial \vartheta} \mathcal{S}_{2,(r+1)} \right]. \quad (74)$$

Like even dimensional case, we again see that we get an additional constraint through entropy current, but one constraint turns out to be weaker. Equilibrium partition function sets $\frac{\partial^2}{\partial \nu \partial \vartheta} f_2(\vartheta, \nu) = 0$ which entropy current fails to do. In Appendix A we will show that introducing torsion remedies this situation.

D. Recap

In this section we summarize the results for the leading derivative order null fluid in mass frame, taking into account the constraints from the equilibrium partition function and the second law of thermodynamics. The constitutive relations for null fluid are given in terms of the fluid variables $\vartheta, \varpi, \nu, u^M$,

$$\begin{aligned} T^{MN} &= Ru^M u^N + 2Eu^{(M} V^{N)} + P P^{MN} + 2\mathcal{E}^{(M} V^{N)} + \Pi^{MN}, \\ J^M &= Qu^M + \Upsilon^M, \end{aligned} \quad (75)$$

where P, R, E, Q are thermodynamic pressure, mass density, energy density and charge density expressed as functions of ϑ, ϖ, ν . These constitutive relations follow the conservation laws,

$$\nabla_M T^{MN} = \mathcal{F}^{NM} J_M, \quad \nabla_M J^M = 0. \quad (76)$$

In odd number of dimensions ($d = 2n - 1$), the form of hydrodynamic frame invariant corrections $\Pi^{MN}, \mathcal{E}^M, \Upsilon^M$ to leading order in derivatives are given as,

$$\begin{aligned} \Pi^{MN} &= -\eta \sigma^{MN} - P^{MN} \zeta \Theta, \\ \mathcal{E}^M &= \kappa_e P^{MN} \partial_N \vartheta + \vartheta \kappa_q P^{MN} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) + \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{e(r)} l_{(r)}^M, \\ \Upsilon^M &= \kappa_q P^{MN} \partial_N \vartheta + \sigma_q P^{MN} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) + \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{q(r)} l_{(r)}^M, \end{aligned} \quad (77)$$

where transport coefficients η (shear viscosity), ζ (bulk viscosity), σ_q (electric conductivity) are some non-negative, κ_e (thermal conductivity) is a nonpositive and κ_q (thermoelectric coefficient) is an arbitrary function of ϑ, ϖ, ν . Parity-odd transport coefficients (Hall conductivities) are however completely determined up to some constants as,

$$\begin{aligned} \tilde{\omega}_{e(r)} &= \vartheta n \left(\vartheta C_{1,(r)} + \frac{E + P - \vartheta \varpi R}{R} C_{2,(r)} - \vartheta \nu C_{2,(r+1)} \right), \\ \tilde{\omega}_{q(r)} &= \vartheta n \left(\frac{Q}{R} C_{2,(r)} - C_{2,(r+1)} \right), \end{aligned} \quad (78)$$

where C 's are some arbitrary constants, and $C_{2,(0)} = 0$. In even number of dimensions ($d = 2n$) however, the corrections are given as,

$$\begin{aligned}
\Pi^{MN} &= -\eta\sigma^{MN} - \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\eta}_{(r)} l_{(r)}^{R(M} \sigma^{N)}{}_R - P^{MN} \left(\zeta\Theta + \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{(r)} l_{(r)} \right), \\
\mathcal{E}^M &= \left(P^{MN} \kappa_\epsilon + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \tilde{\kappa}_{\epsilon(r)} \right) \partial_N \vartheta + \vartheta \left(P^{MN} \kappa_q - \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \tilde{\kappa}_{q(r)} \right) (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) \\
&\quad + \vartheta n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left(\vartheta \partial_N \mathcal{S}_{1,(r)} + \frac{E+P-\vartheta\varpi R}{R} \partial_N \mathcal{S}_{2,(r)} - \vartheta \nu \partial_N \mathcal{S}_{2,(r+1)} \right), \\
\Upsilon^M &= \left(P^{MN} \kappa_q + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \tilde{\kappa}_{q(r)} \right) \partial_N \vartheta + \left(\sigma_q P^{MN} + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \tilde{\sigma}_{q(r)} \right) (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) \\
&\quad + \vartheta n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left(\frac{Q}{R} \partial_N \mathcal{S}_{2,(r)} - \partial_N \mathcal{S}_{2,(r+1)} \right), \tag{79}
\end{aligned}$$

where we have made the following redefinitions with respect to Eq. (48),

$$\begin{aligned}
\tilde{\kappa}_{\epsilon(r)} &\rightarrow \tilde{\kappa}_{\epsilon(r)} + \vartheta n \left[\vartheta \frac{\partial}{\partial \vartheta} \mathcal{S}_{1,(r)} + \frac{E+P-\vartheta\varpi R}{R} \frac{\partial}{\partial \vartheta} \mathcal{S}_{2,(r)} - \vartheta \nu \frac{\partial}{\partial \vartheta} \mathcal{S}_{2,(r+1)} \right], \\
\tilde{\kappa}_{q(r)} &\rightarrow \tilde{\kappa}_{q(r)} + \vartheta n \left[\frac{Q}{R} \frac{\partial}{\partial \vartheta} \mathcal{S}_{2,(r)} - \frac{\partial}{\partial \vartheta} \mathcal{S}_{2,(r+1)} \right]. \tag{80}
\end{aligned}$$

The transport coefficients in parity even sector are same as before; however parity-odd transport coefficients $\tilde{\eta}_{(r)}$ (Hall viscosity), $\tilde{\kappa}_{\epsilon(r)}$ (thermal Hall conductivity), $\tilde{\kappa}_{q(r)}$ (thermoelectric Hall coefficient), $\tilde{\sigma}_{q(r)}$ (electric Hall conductivity), $\mathcal{S}_{1,(r)}$ and $\mathcal{S}_{2,(r)}$ are some arbitrary functions of ϑ , ϖ , ν . Finally $\tilde{\zeta}_{(r)}$ is determined as,

$$\tilde{\zeta}_{(r)} = - \left[\vartheta^2 \frac{\partial P}{\partial E} \frac{\partial}{\partial \vartheta} + \frac{\partial P}{\partial R} \frac{\partial}{\partial \varpi} + \frac{\partial P}{\partial Q} \frac{\partial}{\partial \nu} \right] \mathcal{S}_{2,(r)}. \tag{81}$$

All the constitutive relations satisfy the physical requirements of existence of an equilibrium partition function and entropy current. To leading order in derivatives they are given as, in odd number of dimensions ($d = 2n - 1$),

$$\begin{aligned}
W^{eqb} &= \int \{dx^i\} \sqrt{g} \left[\frac{1}{\vartheta_o} P_o - \sum_{r=0}^{n-1} l_{o(r)}^i \left\{ n \binom{n-1}{r} C_{1,(r)} \tilde{\vartheta} a_i + \binom{n}{r+1} C_{2,(r+1)} A_i \right\} \right], \\
J_s^M &= J_{s(\text{can})}^M + \sum_{r=0}^{n-1} \star \left[\left\{ n \binom{n-1}{r} C_{1,(r)} \vartheta V - \binom{n}{r+1} C_{2,(r+1)} \hat{\mathcal{A}} \right\} \wedge \mathbf{u} \wedge \hat{\mathbf{F}}^{\wedge r} \wedge \hat{\Omega}^{\wedge (n-r-1)} \right]^M, \tag{82}
\end{aligned}$$

and in even number of dimensions ($d = 2n$),

$$\begin{aligned}
W^{eqb} &= \int \{dx^i\} \sqrt{g} \left[\frac{1}{\vartheta_o} P_o - n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{o(r)}^{ij} \mathcal{S}_{o1,(r)} \tilde{\vartheta} \partial_i a_j - \sum_{r=0}^n \binom{n}{r} l_{o(r)} \mathcal{S}_{o2,(r)} \right], \\
J_s^M &= J_{s(\text{can})}^M + \star \left[\left\{ n \sum_{r=0}^{n-1} \binom{n-1}{r} \mathcal{S}_{1,(r)} \mathbf{V} \wedge d\vartheta + \sum_{r=0}^n \binom{n}{r} \mathcal{S}_{2,(r)} \hat{\Omega} \right\} \wedge \mathbf{u} \wedge \hat{\mathcal{F}}^{\wedge r} \wedge \hat{\Omega}^{\wedge (n-r-1)} \right]^M. \tag{83}
\end{aligned}$$

While writing the entropy current from Eq. (67), some total derivative terms have been dropped, as they will have zero divergence. We have included the additional constraints coming from the entropy current analysis while writing the partition function and vice versa. This finishes the discussion of null fluid up to leading order in derivatives in arbitrary number of dimensions. Next, we turn to study the light cone reduction and how to get Galilean fluids via reduction of a null fluid.

IV. LIGHT CONE REDUCTION

We want to study a Galilean system in $(d+1)$ dimensions. So, we essentially want to compactify the V direction as $\mathcal{M}_{(d+2)} = S_V^1 \times \mathcal{M}_{(d+1)}$. But V is null, and thus is transverse to itself, so it is not possible to make such decomposition uniquely. It is therefore convenient to introduce another vector field (we call *time field*) $T = T^M \partial_M$, which can be used to define a unique decomposition $\mathcal{M}_{(d+2)} = S_V^1 \times \mathbb{R}_T^1 \times \mathcal{M}_{(d)}^T$ where $\mathcal{M}_{(d)}$ is the set of vectors transverse to both V and T . The time field T provides a *reference frame* for compactified Galilean theory. We formally define *light cone reduction* as this choice of frame and subsequent compactification.

Using T we can define another null field orthonormal to V ,

$$\bar{V}_{(T)}^M = -\frac{1}{T^N V_N} \left(T^M - \frac{T_R T^R}{2T^S V_S} V^M \right), \quad (84)$$

which satisfies $\bar{V}_{(T)}^M \bar{V}_{(T)M} = 0$, $\bar{V}_{(T)}^M V_M = -1$, and a projection operator transverse to V and \bar{V} ,

$$P_{(T)}^{MN} = G^{MN} + 2\bar{V}_{(T)}^M V^N, \\ \mathcal{M}_{(d)}^T = \{P_{(T)}^{MN} \psi_N : \psi^N \in \mathcal{M}_{(d+2)}\}. \quad (85)$$

Since the choice of T is arbitrary and does not have any physical significance, null theories are invariant under an arbitrary redefinition of $T \rightarrow T'$, which we parametrize as,

$$T^M \rightarrow T'^M = a[T^M - T^N V_N \psi^M], \quad (86)$$

where $\psi^M V_M = 0$ and $a \in \mathbb{R}$. One can check that inverse transformation is simply $a \rightarrow 1/a$, and $\psi^M \rightarrow -\psi^M$. This parametrization has a benefit that under “ T redefinition”,¹⁵

transformation of $\bar{V}_{(T)}^M$ and $P_{(T)}^{MN}$ only depends on $\bar{\psi}^M = P_{(T)}^{MN} \psi_N$,

$$\bar{V}_{(T)}^M \rightarrow \bar{V}_{(T)}^M + \bar{\psi}^M + \frac{1}{2} \bar{\psi}^2 V^M, \\ P_{(T)}^{MN} \rightarrow P_{(T)}^{MN} + 2V^{(M} \bar{\psi}^{N)} + \bar{\psi}^2 V^M V^N, \quad (87)$$

where $\bar{\psi}^2 = P_{(T)}^{MN} \psi_M \psi_N$. Finally, our light cone reduced theory is described on a compactified null background $\mathcal{M}_{(d+2)}$ with isometry $\psi_V = \{V^M, \Lambda_{(V)}\}$, and a time field T^M , modded by diffeomorphisms, gauge transformation and T redefinition. In light cone reduction approach however, we need not worry too much about T redefinition. Since the original theory on null background did not depend on T , so the reduced theory will also be invariant under its redefinition automatically.

A. Newton-Cartan backgrounds by light cone reduction

It is easy to see how the (torsionless¹⁶) Newton-Cartan structure comes out by the light cone reduction of null backgrounds. We identify $\mathcal{M}_{(d+1)}^{\text{NC}} = \mathbb{R}_T^1 \times \mathcal{M}_{(d)}^T$ as the degenerate Newton-Cartan (NC) manifold. Without loss of generality we can choose a basis $x^M = \{x^-, x^\mu\}$ in the original manifold $\mathcal{M}_{(d+2)}$ such that $\psi_V = \{V = \partial_-, \Lambda_{(V)} = 0\}$. x^μ will then provide a basis on NC manifold $\mathcal{M}_{(d+1)}^{\text{NC}}$. This mechanism to generate NC manifold via null reduction was first found in [11] and has been further developed in [5,12,13].

1. Reduction of background fields

We can decompose background fields according to this choice of basis as,

$$V^M = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad V_M = \begin{pmatrix} 0 \\ -n_\mu \end{pmatrix}, \quad \bar{V}_{(T)}^M = \begin{pmatrix} v^\mu \mathcal{B}_\mu \\ v^\mu \end{pmatrix}, \quad \bar{V}_{(T)M} = \begin{pmatrix} -1 \\ \mathcal{B}_\mu \end{pmatrix}, \quad (88)$$

$$P_{(T)MN} = \begin{pmatrix} 0 & 0 \\ 0 & p_{\mu\nu} \end{pmatrix}, \quad P_{(T)}^{MN} = \begin{pmatrix} p^{\nu\rho} \mathcal{B}_\nu \mathcal{B}_\rho & p^{\mu\nu} \mathcal{B}_\nu \\ p^{\mu\nu} \mathcal{B}_\nu & p^{\mu\nu} \end{pmatrix}, \quad (89)$$

such that

$$n_\mu v^\mu = 1, \quad v^\mu p_{\mu\nu} = 0, \quad n_\mu p^{\mu\nu} = 0, \quad p_{\mu\rho} p^{\rho\nu} + n_\mu v^\nu = \delta_\mu^\nu. \quad (90)$$

This is the well-known Newton-Cartan structure. The T -redefinition transformations Eq. (88) becomes Milne boosts on the NC manifold,

¹⁵We would like to note that the T redefinition transformation defined here is parametrized by $d+2$ parameters. A subset of this transformation parametrized by d parameters $\bar{\psi}^M$ will turn into Milne boost transformation upon light cone reduction.

¹⁶Since we are considering only torsionless null backgrounds in this work, upon reduction they will give torsionless Newton-Cartan backgrounds. For a full torsional treatment, look at the companion paper [18].

TABLE II. Leading derivative order data for Galilean fluid.

Null Fluid Data		Newton-Cartan Data
Parity Even		
$\Theta \equiv \nabla_M u^M$ $P^{MN} \partial_N \vartheta, P^{MN} \partial_N \varpi, P^{MN} \partial_N \nu$ $P^{MN} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)$ $\sigma^{MN} \equiv 2P^{MR} P^{NS} \nabla_{(R} u_{S)} - \frac{2}{d} P^{MN} \Theta$		$\Theta \equiv \nabla_\mu u^\mu$ $p^{\mu\nu} \partial_\nu \vartheta, p^{\mu\nu} \partial_\nu \varpi, p^{\mu\nu} \partial_\nu \nu$ $p^{\mu\nu} (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu)$ $\sigma^{\mu\nu} \equiv 2p^{\sigma(\mu} \nabla_{\sigma} u^{\nu)} - \frac{2}{d} p^{\mu\nu} \Theta$
Parity Odd—Odd Dimensions ($d = 2n - 1$)		
$l_{(r)}^{MN} _{r=0}^{n-1} \equiv \star [V \wedge u \wedge \mathcal{F}^{\wedge r} \wedge \Omega^{\wedge(n-r-1)}]^M$		$l_{(r)}^\mu _{r=0}^{n-1} \equiv \star [\beta^{\wedge r} \wedge \omega^{\wedge(n-r-1)}]^\mu$
Parity Odd—Even Dimensions ($d = 2n$)		
$l_{(r)}^{MN} _{r=0}^n \equiv \star [V \wedge u \wedge \mathcal{F}^{\wedge r} \wedge \Omega^{\wedge(n-r)}]$ $l_{(r)}^{MN} \partial_N \vartheta, l_{(r)}^{MN} \partial_N \varpi, l_{(r)}^{MN} \partial_N \nu$ $l_{(r)}^{MN} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)$ $l_{(r)}^{R(M} \sigma_{R}^{N)}$ where, $l_{(r)}^{MN} _{r=0}^{n-1} \equiv \star [V \wedge u \wedge \mathcal{F}^{\wedge r} \wedge \Omega^{\wedge(n-r-1)}]^{MN}$		$l_{(r)} _{r=0}^n \equiv \star [\beta^{\wedge r} \wedge \omega^{\wedge(n-r)}]$ $l_{(r)}^{\mu\nu} \partial_\nu \vartheta, l_{(r)}^{\mu\nu} \partial_\nu \varpi, l_{(r)}^{\mu\nu} \partial_\nu \nu$ $l_{(r)}^{\mu\nu} (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu)$ $l_{(r)}^{\rho(\mu} \sigma_{\rho}^{\nu)}$ $l_{(r)}^{\mu\nu} _{r=0}^{n-1} \equiv \star [\beta^{\wedge r} \wedge \omega^{\wedge(n-r-1)}]^{\mu\nu}$

$$v^\mu \rightarrow v^\mu + \bar{\psi}^\mu, \quad \mathcal{B}_\mu \rightarrow \mathcal{B}_\mu + \bar{\psi}_\mu - \frac{1}{2} n_\mu \bar{\psi}^2, \quad p_{\mu\nu} \rightarrow p_{\mu\nu} - 2n_{(\mu} \bar{\psi}_{\nu)} + n_\mu n_\nu \bar{\psi}^2, \quad (91)$$

from where we can identify v^μ as NC frame velocity. Similarly, the Levi-Civita connection Γ_{MS}^R can be reduced in NC basis as,

$$\begin{aligned} \Gamma_{\mu\nu}^\lambda &= v^\lambda \partial_{(\mu} n_{\nu)} + \frac{1}{2} p^{\lambda\rho} (\partial_\mu p_{\rho\nu} + \partial_\nu p_{\rho\mu} - \partial_\rho p_{\mu\nu}) + \Omega_{(T)\sigma(\mu} n_{\nu)} p^{\sigma\lambda}, \\ \Gamma_{\mu\nu}^- &= p_{\lambda(\mu} \nabla_{\nu)} v^\lambda - \nabla_{(\mu} \mathcal{B}_{\nu)}. \end{aligned} \quad (92)$$

and all other components zero. Here we have identified $\Gamma_{\mu\nu}^\lambda$ as the (torsionless) Newton-Cartan connection and denoted respective covariant derivative by ∇_μ . In Eq. (92) we have treated the derivative of frame velocity $\nabla_\mu v^\nu$ as an independent variable, and have defined the frame vorticity in terms of it as $\Omega_{(T)\mu\nu} = 2p_{\sigma[\nu} \nabla_{\mu]} v^\sigma$. The covariant derivative can be checked to act on the NC structure appropriately,

$$\nabla_\mu n_\nu = 0, \quad \nabla_\mu p^{\rho\sigma} = 0, \quad \nabla_\mu p_{\nu\rho} = -2n_{(\nu} p_{\rho)\sigma} \nabla_\mu v^\sigma. \quad (93)$$

The condition $\mathcal{H} = 0$ implies that $dn = 0$ (where $n = n_\mu dx^\mu$); this is known to be true for the torsionless NC structure. We define the spatial volume element on NC manifold as,

$$\begin{aligned} \varepsilon^{\mu\nu\dots} &= \varepsilon^{MN\mu\nu\dots} V_M \bar{V}_{(T)N} = -\varepsilon^{-\rho\mu\nu\dots} n_\rho, \\ \varepsilon_{\mu\nu\dots} &= p_{\mu\rho} p_{\nu\sigma} \dots \varepsilon^{\rho\sigma\dots}, \end{aligned} \quad (94)$$

and \star as the Hodge duality operation associated with it. The notations and conventions on differential forms can be found in Appendix D. Finally, the only surviving components of the gauge field strength are $\mathcal{F}_{\mu\nu}$, which can be identified as NC gauge field strength. We can further decompose $\mathcal{F}_{\mu\nu}$ and $\Omega_{(T)\mu\nu}$ into,

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= 2e_{[\mu} n_{\nu]} + \beta_{\mu\nu}, \\ \Omega_{(T)\mu\nu} &= -2\alpha_{(T)[\mu} n_{\nu]} + \omega_{(T)\mu\nu}. \end{aligned} \quad (95)$$

All the tensors introduced here are transverse to v^μ . Here e_μ is the electric field while $\beta_{\mu\nu}$ is the dual magnetic field defined with respect to the frame T . Similarly $\alpha_{(T)\mu}$ is the frame acceleration and $\omega_{(T)\mu\nu}$ is the spatial frame vorticity. We can similarly define the fluid acceleration and vorticity as well, which will be used later in Sec. V.

2. Reduction of currents

We decompose the currents of the null theory on $\mathcal{M}_{(d+2)} = S_V^1 \times \mathbb{R}_T^1 \times \mathcal{M}_{(d)}^T$ as,

$$\begin{aligned}
T^{MN} &= \rho \bar{V}_{(T)}^M \bar{V}_{(T)}^N + 2\epsilon_{\text{tot}} V^{(M} \bar{V}_{(T)}^{N)} + 2j_{\rho}^{(M} \bar{V}_{(T)}^{N)} + 2j_{\epsilon}^{(M} V^{N)} \\
&\quad + t^{MN} + \theta_1 V^M V^N, \\
J^M &= q \bar{V}_{(T)}^M + j_q^M + \theta_2 V^M.
\end{aligned} \tag{96}$$

All the introduced tensors are projected along P^{MN} . Note that the redefinitions Eq. (12) can be used to get rid of θ terms. Choosing Newton-Cartan basis we identify, $t^{\mu\nu}$ as stress-energy tensor, ρ, j_{ρ}^{μ} as mass density and current, $\epsilon_{\text{tot}}, j_{\epsilon}^{\mu}$ as energy density and current, and q, j_q^{μ} as charge density and current, as seen by frame T . Under a finite T redefinition Eq. (86) they transform as,

$$\begin{aligned}
\rho &\rightarrow \rho, & j_{\rho}^{\mu} &\rightarrow j_{\rho}^{\mu} - \rho \bar{\psi}^{\mu}, \\
t^{\mu\nu} &\rightarrow t^{\mu\nu} - 2j_{\rho}^{(\mu} \bar{\psi}^{\nu)} + \rho \bar{\psi}^{\mu} \bar{\psi}^{\nu}, \\
\epsilon_{\text{tot}} &\rightarrow \epsilon_{\text{tot}} - j_{\rho}^{\mu} \bar{\psi}_{\mu} + \frac{1}{2} \rho \bar{\psi}^2, \\
j_{\epsilon}^{\mu} &\rightarrow (j_{\epsilon}^{\mu} - \epsilon_{\text{tot}} \bar{\psi}^{\mu}) - (t^{\mu\nu} - \bar{\psi}^{\mu} j_{\rho}^{\nu}) \bar{\psi}_{\nu} + \frac{1}{2} \bar{\psi}^2 (j_{\rho}^{\mu} - \rho \bar{\psi}^{\mu}), \\
q &\rightarrow q, & j_q^{\mu} &\rightarrow j_q^{\mu} - q \bar{\psi}^{\mu}.
\end{aligned} \tag{97}$$

These transformations can be immediately identified as Milne boost transformation of Galilean theories. Note that we have used the same time field (reference frame) to decompose the currents/densities as well as the background fields. It is sometimes required to define background in one reference frame (e.g. lab frame) but currents and densities in some other reference frame (e.g. comoving frame). One

can merely perform a Milne boost on various quantities noted above and gain the desired result.

3. Reduction of Ward identities

On the decomposition $\mathcal{M}_{(d+2)} = S_V^1 \times \mathbb{R}_T^1 \times \mathcal{M}_{(d)}^T$, background field content is $V_M, \bar{V}_{(T)M}, P_{(T)MN}$ and \mathcal{A}_M , so any physical theory should be described by a partition function $W[V_M, \bar{V}_{(T)M}, P_{(T)MN}, \mathcal{A}_M]$. Using the current redefinitions Eq. (11) for null backgrounds, we can parametrize the variation of partition function as,

$$\begin{aligned}
\delta W &= \int \{dx^M\} \sqrt{-G} \left[(\epsilon_{\text{tot}} \bar{V}_{(T)}^M + j_{\epsilon}^M) \delta V_M \right. \\
&\quad + (\rho \bar{V}_{(T)}^M + j_{\rho}^M) \delta \bar{V}_{(T)M} \\
&\quad + \left(p^M \bar{V}_{(T)}^N + \frac{1}{2} t^{MN} \right) \delta P_{(T)MN} \\
&\quad \left. + (q \bar{V}_{(T)}^M + j_q^M) \delta \mathcal{A}_M \right].
\end{aligned} \tag{98}$$

The same partition function should also be gained by directly reducing the relativistic partition function Eq. (8). This will render the quantities in above partition function to be same as the ones defined in Eq. (96), and in addition $p^M = j_{\rho}^M$. The latter constraint is Ward identity of T redefinition, i.e. can be gained by demanding partition function Eq. (98) to be invariant under T redefinition.

Choosing NC basis the partition function variation Eq. (98) can be decomposed to,

$$\delta W = \int \{dx^M\} \sqrt{\det(p_{\mu\nu} + n_{\mu} n_{\nu})} \left[-(\epsilon_{\text{tot}} v^{\mu} + j_{\epsilon}^{\mu}) \delta n_{\mu} + (\rho v^{\mu} + j_{\rho}^{\mu}) \delta \mathcal{B}_{\mu} + \left(p^{\mu} v^{\nu} + \frac{1}{2} t^{\mu\nu} \right) \delta p_{\mu\nu} + (q v^{\mu} + j_q^{\mu}) \delta \mathcal{A}_{\mu} \right]. \tag{99}$$

Symmetry data of the light cone reduced theory is,

$$\psi_{\xi}^{\text{NC}} = \{\xi^-, \xi_- = -n_{\mu} \xi^{\mu}, \bar{\xi}^{\mu} = p_{\nu}^{\mu} \xi^{\nu}, \Lambda_{(\xi)}, \bar{\psi}^{\mu} = p_{\nu}^{\mu} \psi^{\nu}\}, \tag{100}$$

where we identify ξ^- as mass parameter, ξ_- as time translation parameter, $\bar{\xi}^{\mu}$ as space translation parameter, $\Lambda_{(\xi)}$ as gauge parameter, and $\bar{\psi}^{\mu}$ as Milne boost parameter. The respective Ward identities can be found using Eq. (99) or directly reducing the null Ward identities Eq. (9),

$$\begin{aligned}
\text{Mass Conservation:} & \quad \nabla_{\mu} (\rho v^{\mu} + j_{\rho}^{\mu}) = 0, \\
\text{Energy Conservation:} & \quad \nabla_{\mu} (\epsilon_{\text{tot}} v^{\mu} + j_{\epsilon}^{\mu}) = e_{\mu} j_q^{\mu} - (v^{\mu} p_{\nu} + t^{\mu}_{\nu}) \nabla_{\mu} v^{\nu}, \\
\text{Momentum Conservation:} & \quad \nabla_{\mu} (v^{\mu} p^{\nu} + t^{\mu\nu}) = [q e^{\nu} + \beta^{\nu}_{\mu} j_q^{\mu}] - (\rho v^{\mu} + j_{\rho}^{\mu}) \nabla_{\mu} v^{\nu}, \\
\text{Charge Conservation:} & \quad \nabla_{\mu} (q v^{\mu} + j_q^{\mu}) = 0, \\
\text{Milne Identity:} & \quad j_{\rho}^{\mu} = p^{\mu}.
\end{aligned} \tag{101}$$

First terms in the right-hand side (RHS) of energy and momentum conservation equations are work done and Lorentz force due to electromagnetic fields, while the last terms are pseudoenergy and pseudoforce due to spacetime dependence of the frame velocity. As we already mentioned, the Milne identity is trivial in theories obtained by reduction.

All these results here have been mentioned in Newton-Cartan notation, which is a nice covariant formalism for Galilean physics. However for familiarity and to build intuition, we have given all these results in conventional noncovariant notation as well in Appendix B.

B. Equilibrium on Newton-Cartan backgrounds

From the perspective of Galilean theories, equilibrium is defined by a preferred reference frame (or time field) K with respect to which system does not evolve in time. This can be achieved by reducing null theories at equilibrium, and identify the timelike isometry ψ_K with preferred reference frame in the Galilean theory. Hence the variation of equilibrium partition function in local rest of reference frame K is essentially same as the null fluid Eq. (18) written in terms of Galilean quantities,

$$\begin{aligned} \delta W^{eqb} = \int \{dx^i\} \sqrt{g} & \left[\epsilon_o \frac{1}{g_o^2} \delta \vartheta_o \right. \\ & + \left(-\frac{1}{g_o} j_{oe}^i + \varpi_o j_{op}^i + \nu_o j_{oq}^i \right) \frac{1}{g_o} \tilde{\vartheta} \delta a_i \\ & + \frac{1}{2g_o} t_o^{ij} \delta g_{ij} + \left(\rho \delta \varpi_o + \frac{1}{g_o} j_{op}^i \delta B_i \right) \\ & \left. + \left(q \delta \nu_o + \frac{1}{g_o} j_{oq}^i \delta A_i \right) \right], \end{aligned} \quad (102)$$

and hence,

$$\begin{aligned} \rho_o &= \frac{\delta W^{eqb}}{\delta \varpi_o}, & j_{op}^i &= g_o \frac{\delta W^{eqb}}{\delta B_i}, & t_o^{ij} &= 2g_o \frac{\delta W^{eqb}}{\delta g_{ij}}, \\ q_o &= \frac{\delta W^{eqb}}{\delta \nu_o}, & j_{oq}^i &= g_o \frac{\delta W^{eqb}}{\delta A_i}, & \epsilon_o &= g_o^2 \frac{\delta W^{eqb}}{\delta \vartheta_o}, \\ j_{oe}^i - \varpi_o g_o j_{op}^i - \nu_o g_o j_{oq}^i &= -g_o e^\Phi \frac{\delta W^{eqb}}{\delta a_i}. \end{aligned} \quad (103)$$

Here all the observables are defined as seen by reference frame K , and are denoted by a subscript $_o$. These will reduce to the expected relations Eq. (3) in flat space, i.e.

$$\begin{aligned} \vartheta_o &= 1, & g_{ij} &= \delta_{ij}, \\ \varpi_o &= \nu_o = A_i = B_i = a_i = 0. \end{aligned} \quad (104)$$

Since we have fixed the T -redefinition symmetry by choosing a preferred reference frame K , the corresponding EOM does not show up. Consequently, momentum current p^μ does not appear in the partition function, and can be found by using the missed EOM. In equilibrium configuration, null fluid and Galilean fluid have same field content and symmetries, so we expect the equilibrium partition function to also be the same. To ideal order Eq. (25) it will identify ρ, ϵ, q with thermodynamic functions R, E, Q , and hence will give physical interpretation to thermodynamics

of null theories in terms of Galilean physics. In hydrodynamic description, at further derivative orders also, it will give physical interpretation to various transport coefficients and constraints of null fluid.

V. GALILEAN HYDRODYNAMICS

Having discussed the light cone reduction of generic null theories in the last section, we can straight away perform light cone reduction of null fluids in Sec. III and hope to get Galilean fluids. If we look at Eq. (32) closely, it is already nicely organized in local rest frame of the fluid (defined by $\bar{V}^M = u^M$). One just needs to apply a T -redefinition (Milne boost) to it with $\psi^M = -u^M$ to get densities and currents in a generic reference frame,

$$\begin{aligned} j_\rho^\mu &= p^\mu = \rho \bar{u}^\mu + \zeta_\rho^\mu, \\ t^{\mu\nu} &= \rho \bar{u}^\mu \bar{u}^\nu + P p^{\mu\nu} + \pi^{\mu\nu} + 2\bar{u}^{(\mu} \zeta_{\rho}^{\nu)}, \\ j_q^\mu &= q \bar{u}^\mu + \zeta_q^\mu, \\ \epsilon_{\text{tot}} &= \epsilon + \frac{1}{2} \rho \bar{u}^2 + \zeta_\rho^\mu \bar{u}_\mu, \\ j_\epsilon^\mu &= (\epsilon_{\text{tot}} + P) \bar{u}^\mu + \zeta_\epsilon^\mu + \pi^{\mu\nu} \bar{u}_\nu + \frac{1}{2} \zeta_\rho^\mu \bar{u}^2, \end{aligned} \quad (105)$$

where we have identified,

$$\begin{aligned} \rho &= \mathcal{R}, & q &= \mathcal{Q}, & \epsilon &= \mathcal{E}, & \bar{u}^\mu &= u^\nu p_\nu^\mu, \\ \zeta_\rho^\mu &= \mathbb{R}^\mu, & \zeta_q^\mu &= \mathbb{J}^\mu, & \zeta_\epsilon^\mu &= \mathbb{E}^\mu, \\ \pi^{\mu\nu} &= (\mathcal{P} - P) p^{\mu\nu} + \mathbb{T}^{\mu\nu}, \end{aligned} \quad (106)$$

and $\bar{u}^2 = \bar{u}^\mu \bar{u}_\mu$. Similarly, entropy current of the Galilean fluid can be found to be,

$$\begin{aligned} s &= \frac{\epsilon + P}{g} - \varpi \rho - \nu q - \Upsilon_{s-}, \\ j_s^\mu &= s \bar{u}^\mu + \frac{1}{g} \zeta_\epsilon^\mu - \varpi \zeta_\rho^\mu - \nu \zeta_q^\mu + \Upsilon_s^\nu p_\nu^\mu, \end{aligned} \quad (107)$$

which follows the second law of thermodynamics

$$\nabla_\mu (s v^\mu + j_s^\mu) \geq 0. \quad (108)$$

Choosing “mass frame” for the null fluid, which is the most natural frame from a Galilean perspective, will switch off $\zeta_\rho^\mu = \mathbb{R}^\mu$, and hence Galilean mass current will not undergo any dissipation. The identifications for mass frame are given by Eq. (106), and can be read out in terms of frame invariants as,

$$\begin{aligned} \rho &= R, & q &= Q, & \epsilon &= E, & s &= S - \Upsilon_{s-}, & \bar{u}^\mu &= u^\nu p_\nu^\mu, \\ \zeta_\rho^\mu &= 0, & \zeta_q^\mu &= \Upsilon^\mu, & \zeta_\epsilon^\mu &= \mathcal{E}^\mu, & \pi^{\mu\nu} &= \Pi^{\mu\nu}, \end{aligned} \quad (109)$$

and in turn the constitutive relations become,

$$\begin{aligned}
j_\rho^\mu &= p^\mu = R\bar{u}^\mu, & t^{\mu\nu} &= R\bar{u}^\mu\bar{u}^\nu + Pp^{\mu\nu} + \pi^{\mu\nu}, \\
j_q^\mu &= Q\bar{u}^\mu + \varsigma_q^\mu, & \hat{\epsilon}_{\text{tot}} &= E + \frac{1}{2}R\bar{u}^2, \\
j_\epsilon^\mu &= \left(E + P + \frac{1}{2}R\bar{u}^2\right)\bar{u}^\mu + \varsigma_\epsilon^\mu + \pi^{\mu\nu}\bar{u}_\nu, \\
\hat{s} &= S - \Upsilon_{s-}, \\
j_s^\mu &= S\bar{u}^\mu + \frac{1}{\vartheta}\varsigma_\epsilon^\mu - \nu\varsigma_q^\mu + \Upsilon_s^\nu p_\nu^\mu - \Upsilon_{s-}\bar{u}^\mu. \tag{110}
\end{aligned}$$

These are the standard Galilean constitutive relations, written in Newton-Cartan basis. We will present all these expressions in conventional noncovariant basis in

Appendix B for the benefit of readers not comfortable with the Newton-Cartan formalism.

Having obtained the general picture, we can now deduce constitutive relations for a Galilean fluid up to leading order in derivative expansion, using the corresponding null fluid results in Sec. III D. In Table II we have mentioned the light cone reduction of all the leading order data to get Newton-Cartan data. Having done so, the rest of the algebra is essentially trivial. In the following we will work in the mass frame explicitly.

Even dimensional Galilean fluids: Using reduction of data enlisted in Table II, we can read out the even dimensional ($d = 2n - 1$) constitutive relations from Sec. III D,

$$\begin{aligned}
\pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - p^{\mu\nu}\zeta\Theta, \\
\varsigma_q^\mu &= \kappa_q p^{\mu\nu}\partial_\nu\vartheta + \sigma_q p^{\mu\nu}(\mathcal{F}_{\nu\rho}u^\rho - \vartheta\partial_\nu\nu) + \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{q(r)} l_{(r)}^\mu, \\
\varsigma_\epsilon^\mu &= \kappa_\epsilon p^{\mu\nu}\partial_\nu\vartheta + \vartheta\kappa_q p^{\mu\nu}(\mathcal{F}_{\nu\rho}u^\rho - \vartheta\partial_\nu\nu) + \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{\epsilon(r)} l_{(r)}^\mu, \tag{111}
\end{aligned}$$

where transport coefficients η (shear viscosity), ζ (bulk viscosity), σ_q (electric conductivity) are some non-negative, κ_ϵ (thermal conductivity) is a nonpositive and κ_q (thermoelectric coefficient) is an arbitrary function of ϑ, ϖ, ν . Parity-odd transport coefficients (Hall conductivities) are however completely determined up to some constants as,

$$\begin{aligned}
\tilde{\omega}_{\epsilon(r)} &= \vartheta n \left(\vartheta C_{1,(r)} + \frac{E + P - \vartheta\varpi R}{R} C_{2,(r)} - \vartheta\nu C_{2,(r+1)} \right), \\
\tilde{\omega}_{q(r)} &= \vartheta n \left(\frac{Q}{R} C_{2,(r)} - C_{2,(r+1)} \right), \tag{112}
\end{aligned}$$

where C 's are some arbitrary constants, and $C_{2,(0)} = 0$. As a special case one can obtain the 4 dimensional ($d = 3$) results which will only affect the parity odd sector,

$$\begin{aligned}
\pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - p^{\mu\nu}\zeta\Theta, \\
\varsigma_q^\mu &= \kappa_q p^{\mu\nu}\partial_\nu\vartheta + \sigma_q p^{\mu\nu}(\mathcal{F}_{\nu\rho}u^\rho - \vartheta\partial_\nu\nu) + \tilde{\omega}_{q(0)}\omega^\mu + \tilde{\omega}_{q(1)}B^\mu, \\
\varsigma_\epsilon^\mu &= \kappa_\epsilon p^{\mu\nu}\partial_\nu\vartheta + \vartheta\kappa_q p^{\mu\nu}(\mathcal{F}_{\nu\rho}u^\rho - \vartheta\partial_\nu\nu) + \tilde{\omega}_{\epsilon(0)}\omega^\mu + \tilde{\omega}_{\epsilon(1)}B^\mu, \tag{113}
\end{aligned}$$

where,

$$\omega^\mu = l_{(0)}^\mu = \frac{1}{2}\epsilon^{\nu\rho\mu}\omega_{\nu\rho}, \quad B^\mu = l_{(1)}^\mu = \frac{1}{2}\epsilon^{\nu\rho\mu}\beta_{\nu\rho}, \tag{114}$$

are vorticity and gauge magnetic fields respectively, and,

$$\begin{aligned}
\tilde{\omega}_{\epsilon(0)} &= 2\vartheta(\vartheta C_{1,(0)} - \vartheta\nu C_{2,(1)}), \\
\tilde{\omega}_{\epsilon(1)} &= 2\vartheta \left(\vartheta C_{1,(1)} + \frac{E + P - \vartheta\varpi R}{R} C_{2,(1)} - \vartheta\nu C_{2,(2)} \right), \\
\tilde{\omega}_{q(0)} &= -2\vartheta C_{2,(1)}, \quad \tilde{\omega}_{q(1)} = 2\vartheta \left(\frac{Q}{R} C_{2,(1)} - C_{2,(2)} \right). \tag{115}
\end{aligned}$$

e. Odd dimensional Galilean fluids: Using reduction of data enlisted in Table II, we can read out the odd dimensional ($d = 2n$) constitutive relations from Sec. III D,

$$\begin{aligned}
\pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\eta}_{(r)} l_{(r)}^{\rho(\mu} \sigma^{\nu)\sigma} p_{\rho\sigma} - p^{\mu\nu} \left(\zeta\Theta + \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{(r)} l_{(r)} \right), \\
\varsigma_q^\mu &= \left(p^{\mu\nu} \kappa_q + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{\mu\nu} \tilde{\kappa}_{q(r)} \right) \partial_\nu \vartheta + \left(p^{\mu\nu} \sigma_q + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{\mu\nu} \tilde{\sigma}_{q(r)} \right) (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu) \\
&\quad + \vartheta n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{\mu\nu} \left(\frac{Q}{R} \partial_\nu \mathcal{S}_{2,(r)} - \partial_\nu \mathcal{S}_{2,(r+1)} \right), \\
\varsigma_\epsilon^\mu &= \left(p^{\mu\nu} \kappa_\epsilon + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{\mu\nu} \tilde{\kappa}_{\epsilon(r)} \right) \partial_\nu \vartheta + \vartheta \left(p^{\mu\nu} \kappa_q - \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{\mu\nu} \tilde{\kappa}_{q(r)} \right) (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu) \\
&\quad + \vartheta n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{\mu\nu} \left(\vartheta \partial_\nu \mathcal{S}_{1,(r)} + \frac{E + P - \vartheta \varpi R}{R} \partial_\nu \mathcal{S}_{2,(r)} - \vartheta \partial_\nu \mathcal{S}_{2,(r+1)} \right). \tag{116}
\end{aligned}$$

The transport coefficients in parity even sector are same as before; however parity-odd transport coefficients $\tilde{\eta}_{(r)}$ (Hall viscosity), $\tilde{\kappa}_{\epsilon(r)}$ (thermal Hall conductivity), $\tilde{\kappa}_{q(r)}$ (thermoelectric Hall coefficient), $\tilde{\sigma}_{q(r)}$ (electric Hall conductivity), $\mathcal{S}_{1,(r)}$ and $\mathcal{S}_{2,(r)}$ are some arbitrary functions of ϑ , ϖ , ν . Finally $\tilde{\zeta}_{(r)}$ is determined as,

$$\tilde{\zeta}_{(r)} = - \left[\vartheta^2 \frac{\partial P}{\partial E} \frac{\partial}{\partial \vartheta} + \frac{\partial P}{\partial R} \frac{\partial}{\partial \varpi} + \frac{\partial P}{\partial Q} \frac{\partial}{\partial \nu} \right] \mathcal{S}_{2,(r)}. \tag{117}$$

As a special case we would like to write down the 3 dimensional results,

$$\begin{aligned}
\pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - \tilde{\eta} \epsilon^{\rho(\mu} \sigma^{\nu)\sigma} p_{\rho\sigma} - p^{\mu\nu} (\zeta\Theta + \tilde{\zeta}_\omega \omega + \tilde{\zeta}_B B), \\
\varsigma_q^\mu &= (p^{\mu\nu} \kappa_q + \epsilon^{\mu\nu} \tilde{\kappa}_q) \partial_\nu \vartheta + (p^{\mu\nu} \sigma_q + \epsilon^{\mu\nu} \tilde{\sigma}_q) (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu) + \epsilon^{\mu\nu} \left(\frac{Q}{R} \vartheta \partial_\nu \mathcal{S}_{2,(0)} - \vartheta \partial_\nu \mathcal{S}_{2,(1)} \right), \\
\varsigma_\epsilon^\mu &= (p^{\mu\nu} \kappa_\epsilon + \epsilon^{\mu\nu} \tilde{\kappa}_\epsilon) \partial_\nu \vartheta + \vartheta (p^{\mu\nu} \kappa_q - \epsilon^{\mu\nu} \tilde{\kappa}_q) (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu) \\
&\quad + \vartheta \epsilon^{\mu\nu} \left(\vartheta \partial_\nu \mathcal{S}_{1,(0)} + \frac{E + P - \vartheta \varpi R}{R} \partial_\nu \mathcal{S}_{2,(0)} - \vartheta \partial_\nu \mathcal{S}_{2,(1)} \right), \tag{118}
\end{aligned}$$

where,

$$\omega = l_{(0)} = \frac{1}{2} \epsilon^{\mu\nu} \omega_{\mu\nu}, \quad B = l_{(1)} = \frac{1}{2} \epsilon^{\mu\nu} \mathcal{F}_{\mu\nu}, \tag{119}$$

are again the vorticity and gauge magnetic fields and we have renamed $\tilde{\sigma}_q = \tilde{\sigma}_{q(0)}$, $\tilde{\kappa}_\epsilon = \tilde{\kappa}_{\epsilon(0)}$, $\tilde{\kappa}_q = \tilde{\kappa}_{q(0)}$, $\tilde{\eta} = \tilde{\eta}_{(0)}$, $\tilde{\zeta}_\omega = \tilde{\zeta}_{(0)}$, $\tilde{\zeta}_B = \tilde{\zeta}_{(1)}$. The 3 dimensional Galilean fluid was also studied by [2], however we find certain discrepancies in their and our results. A detailed comparison has been provided in Appendix C.

Before closing this discussion we would like to note that [6] also constructed an equilibrium partition function and entropy current for an uncharged 3 and 4 dimensional Galilean fluid, and used it to constraint the respective constitutive relations. By switching off the charge sector

and setting $d = 2, 3$ we see that we trivially recover their results.

This finishes our discussion of (nonanomalous) constitutive relations of a Galilean fluid up to leading order in derivatives in arbitrary number of dimensions, obtained by light cone reduction of a null fluid. Unlike the hydrodynamic reductions before this work [14,15], there is no nontrivial mapping between the relativistic (null) fluid and the Galilean fluid. In fact term by term, null fluid constitutive relations are same as Galilean constitutive relations. The same is true for thermodynamics, entropy current and the equilibrium partition function as well. We deduce that we can see null fluid as Galilean fluid written in extended space representation. Many aspects of it are already hinted by extended space construction of [2]. In the next section we extend this approach to study effect of $U(1)$ anomaly on fluid transport.

VI. ANOMALIES

Up to this point we have studied hydrodynamics on nonanomalous null/Galilean backgrounds. In this section we want to explore if the null background construction can also be used to introduce $U(1)$ anomaly in Galilean theories.¹⁷ Later we will find how constraints of Galilean fluid modify in the presence of anomalies. We use the anomaly inflow mechanism of usual relativistic backgrounds to achieve this goal, with appropriate modifications due to the null structure of the background. In this work we will only be interested in classifying the possible $U(1)$ anomalies in Galilean theories motivated from their relativistic counterpart. The field theoretic interpretation of these anomalies is not yet clear to us. It is interesting to note however that the same anomalies were also found in the path integral study of Lifshitz fermions in [20].

Consider a *bulk* manifold $\mathcal{B}_{(d+3)}$, on whose boundary $\mathcal{M}_{(d+2)}$ our theory of interest, i.e. null fluid lives. Indices on $\mathcal{B}_{(d+3)}$ are denoted with a bar \bar{M}, \bar{N}, \dots . We define $\mathcal{B}_{(d+3)}$ also as a null background, with respective fields $\mathcal{A}_{\bar{M}}$, $G_{\bar{M}\bar{N}}$ and a compatible null isometry¹⁸ $\psi_V = \{V = V^{\bar{M}}\partial_{\bar{M}}, \Lambda_{(V)} = 0\}$, such that transverse components of all these fields vanish at boundary. We can define respective fields on $\mathcal{M}_{(d+2)}$ by pulling back the bulk fields, which gives it a null background structure.

We start with the assumption that full theory on $\mathcal{B}_{(d+3)} \cup \mathcal{M}_{(d+2)}$ described by a partition function \mathcal{W} is gauge invariant. Most generic such partition function can be decomposed into a bulk and a boundary piece,

$$\mathcal{W} = W[\mathcal{M}_{(d+2)}] + W_{\text{bulk}}[\mathcal{B}_{(d+3)}], \quad (120)$$

which individually are not gauge invariant. Here W is the partition function of the boundary null theory which is *anomalous*, i.e. is not gauge invariant. W_{bulk} on the other hand is a pure bulk piece whose gauge variation must be a boundary term. While constructing W_{bulk} out of background fields, we can let go of any terms which are gauge invariant up to a total derivative (we can always redefine W to absorb this total derivative term at the boundary), as they will not induce any anomalies in the boundary theory. Hence allowed W_{bulk} can be written as integration of a full rank form,

¹⁷Galilean anomalies considered in [19] are different than what we are considering in this paper, because our background field content does not match that of [19] after reduction (we have chosen $\mathcal{A}_- = 0$). A detailed comparison of these issues along with an extension to non-Abelian and gravitational anomalies will shortly appear in a companion paper [18].

¹⁸We would like to mention that this construction only seems to work when we set $\Lambda_{(V)} = 0$. We give more reasoning in this regard in a companion paper.

$$W_{\text{bulk}} = \int_{\mathcal{B}_{(d+3)}} \mathbf{I}^{(d+3)}, \quad (121)$$

such that $\mathbf{I}^{(d+3)}$ has an exact gauge variation $\delta_{\Lambda_{(\xi)}} \mathbf{I}^{(d+3)} = dG_{(\xi)}$, and it must not be symmetry invariant up to an exact form.¹⁹ In usual relativistic theories, $\mathbf{I}^{(d+3)}$ can only be written in odd bulk dimensions ($d = 2n - 2$), and is given by the *Chern-Simons* form $\mathbf{I}_{CS}^{(2n+1)}$,

$$\mathbf{I}_{CS}^{(2n+1)} = C^{(2n)} \mathcal{A} \wedge \mathcal{F}^{\wedge n}. \quad (122)$$

However for null backgrounds, this term identically vanishes, as it is a full rank form but does not have any component along V . We are therefore forced to modify $\mathbf{I}^{(d+3)}$, by adding some term which has nonvanishing component along V . We do it by choosing an arbitrary time-field T and use it to define a conjugate null field $\bar{V}_{(T)}$. Now we can define an analogue of Chern-Simons form, but in even bulk dimensions ($d = 2n - 1$),

$$\mathbf{I}^{(2n+2)} = -C^{(2n)} \bar{V}_{(T)} \wedge \mathcal{A} \wedge \mathcal{F}^{\wedge n}. \quad (123)$$

We need to check if it fits our requirements. We will leave it for the readers to convince themselves that this expression cannot be transformed into a symmetry invariant term by adding an exact form. For the other criteria we need to compute its gauge variation,

$$\begin{aligned} \delta_{\Lambda_{(\xi)}} \mathbf{I}^{(2n+2)} &= C^{(2n)} d\Lambda_{(\xi)} \wedge \bar{V}_{(T)} \wedge \mathcal{F}^{\wedge n} \\ &= d(C^{(2n)} \Lambda_{(\xi)} \bar{V}_{(T)} \wedge \mathcal{F}^{\wedge n}) \\ &\quad - C^{(2n)} \Lambda_{(\xi)} d\bar{V}_{(T)} \wedge \mathcal{F}^{\wedge n}. \end{aligned} \quad (124)$$

The last term vanishes as it has again no component along V , thus we verify that gauge variation of $\mathbf{I}^{(2n+2)}$ is a boundary term. It is important to note that while we have used $\bar{V}_{(T)}$ to define $\mathbf{I}^{(2n+2)}$, it is invariant under T -redefinition. One can check there does not exist any other term which meets these criteria. Hence contrary to usual relativistic backgrounds, here we can only define $\mathbf{I}^{(2n+2)}$ in even bulk dimensions. It means that only odd dimensional null backgrounds (the one at the boundary) and hence even dimensional Galilean backgrounds (that we get by reducing the boundary null theory) can be anomalous, which is what we expect.

A. Anomalous ward identities

In this section, we present the modified Ward identities in presence of $U(1)$ anomaly. In presence of anomalies, variation of boundary partition function W generates *consistent currents* T_{cons}^{MN} and J_{cons}^M which are not gauge

¹⁹ $\mathbf{I} \neq \mathbf{I}' + dX$, for some gauge invariant \mathbf{I}' .

invariant. Varying $I^{(2n+2)}$ we can now write down variation of the full partition function,

$$\delta\mathcal{W} = \int_{\mathcal{B}_{(2n+2)}} (n+1)C^{(2n)}\delta\mathcal{A} \wedge \bar{V}_{(T)} \wedge \mathcal{F}^{\wedge n} + \int \{dx^M\}\sqrt{G}\left[\frac{1}{2}T^{MN}\delta G_{MN} + J^M\delta\mathcal{A}_M\right], \quad (125)$$

where we have defined the covariant currents,

$$T^{MN} = T_{\text{cons}}^{MN}, \quad J^M = J_{\text{cons}}^M + nC^{(2n)}\star[\bar{V}_{(T)} \wedge \mathcal{A} \wedge \mathcal{F}^{\wedge(n-1)}]^M. \quad (126)$$

Since the full partition function is gauge invariant, and we see that the bulk piece is manifestly gauge invariant, therefore covariant currents must also be gauge invariant. Demanding \mathcal{W} to be gauge invariant we can get the Ward identities in the boundary theory,

$$\nabla_M T^{MN} = \mathcal{F}^{NM} J_M, \quad \nabla_M J^M = -(n+1)C^{(2n)}\star[\bar{V}_{(T)} \wedge \mathcal{F}^{\wedge n}]. \quad (127)$$

We observe that the system exhibits $U(1)$ anomaly. Performing null reduction of these Ward identities one can get the anomalous Galilean Ward identities,

Mass Conservation:	$\nabla_\mu (\rho v^\mu + j_\rho^\mu) = 0,$
Energy Conservation:	$\nabla_\mu (\epsilon_{\text{tot}} v^\mu + j_\epsilon^\mu) = e_\mu j_q^\mu - (v^\mu p_\mu + t_\nu^\mu) \nabla_\mu v^\nu,$
Momentum Conservation:	$\nabla_\mu (v^\mu p^\nu + t^{\mu\nu}) = [qe^\nu + \beta_\mu^\nu j_q^\mu] - (\rho v^\mu + j_\rho^\mu) \nabla_\mu v^\nu,$
Charge Conservation:	$\nabla_\mu (qv^\mu + j_q^\mu) = -\frac{(n+1)}{2n}C^{(2n)}\epsilon_\uparrow^{\mu\nu\dots}\mathcal{F}_{\mu\nu} \wedge \dots,$
Milne Identity:	$j_\rho^\mu = p^\mu.$

(128)

Here $\epsilon_\uparrow^{\mu\nu\dots} = \epsilon_\uparrow^{M\nu\dots}\bar{V}_{(T)M}$ is the raised NC volume element. Comparing to Eq. (101) one can check that all Ward identities except the charge conservation remain nonanomalous. Interestingly, we observe that these anomalies are the same as found by [20] for Lifshitz fermions using path integral methods.

B. Anomalous equilibrium partition function

In our earlier discussion on equilibrium in Sec. II B, we wrote the most generic equilibrium partition function as a gauge invariant scalar. Now we need to modify this partition function appropriately with a gauge noninvariant piece to account for anomaly. We decompose the equilibrium partition function into,

$$W^{eqb} = W_{\text{cons}}^{eqb} + W_{\text{anom}}^{eqb}. \quad (129)$$

Here W_{cons}^{eqb} is the most generic gauge invariant partition function which can be written out of background fields, which has been discussed thoroughly in preceding sections. W_{anom}^{eqb} on the other hand is completely determined in terms of anomaly coefficient $C^{(2n)}$. We suggest its explicit form to be,

$$W_{\text{anom}}^{eqb} = - \int_{\mathcal{M}_{(2n+1)}} nC^{(2n)}\partial_o\nu_o V \wedge \bar{V}_{(K)} \wedge \mathcal{A} \wedge (\mathcal{F}^{\wedge(n-1)} + \frac{1}{2}(n-1)d(\partial_o\nu_o V) \wedge \mathcal{F}^{\wedge(n-2)}), \quad (130)$$

which generates correct anomalies. Recall that we are allowed to use any arbitrary time field T to specify anomalies; in equilibrium configuration $T = K$ is the most natural choice. Performing variation ($dV = 0$) of W_{anom}^{eqb} ,

$$\delta W_{\text{anom}}^{eqb} = - \int_{\mathcal{M}_{(2n+1)}} \frac{1}{2}n(n+1)C^{(2n)}\partial_o^2\nu_o^2\delta V \wedge V \wedge \bar{V}_{(K)} \wedge \mathcal{F}^{\wedge(n-1)} - \int_{\mathcal{M}_{(2n+1)}} nC^{(2n)}\delta\mathcal{A} \wedge \{(n+1)\partial_o\nu_o V \wedge \bar{V}_{(K)} \wedge \mathcal{F}^{\wedge(n-1)} - \bar{V}_{(K)} \wedge \mathcal{A} \wedge \mathcal{F}^{\wedge(n-1)}\}, \quad (131)$$

and using Eq. (126), we can find the covariant anomalous currents at equilibrium,

$$J_{o,\text{anom}}^M = n(n+1)C^{(2n)}\partial_o\nu_o\star[V \wedge \bar{V}_{(K)} \wedge \mathcal{F}^{\wedge(n-1)}]^M, \quad T_{o,\text{anom}}^{MN} = n(n+1)C^{(2n)}\partial_o^2\nu_o^2\star[V \wedge \bar{V}_{(K)} \wedge \mathcal{F}^{\wedge(n-1)}]^{(M}V^{N)}. \quad (132)$$

One can check that these currents identically satisfy the anomalous conservation equations. Note that these currents are also to be supplemented with the nonanomalous pieces discussed in previous sections. In the local rest of reference frame K , the equilibrium partition function can be expressed in Kaluza-Klein notation,

TABLE III. Leading derivative order data for Galilean fluid (Noncovariant).

Newton-Cartan Data	Noncovariant Data
Parity Even	
$\nabla_\mu u^\mu$	$\nabla_i \bar{u}^i - \frac{1}{2}(e^\Phi - a_i \bar{u}^i) g_{ij} \partial_t g^{ij} - a_i \partial_t \bar{u}^i$
$p^{\mu\nu} \partial_\nu \vartheta, p^{\mu\nu} \partial_\nu \varpi, p^{\mu\nu} \partial_\nu \nu$	$\nabla^i \vartheta - a^i \partial_t \vartheta, \nabla^i \varpi - a^i \partial_t \varpi, \nabla^i \nu - a^i \partial_t \nu$
$p^{\mu\nu} (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu)$	$e^i + \beta^{ij} \bar{u}_j - \vartheta \nabla^i \nu + \vartheta a^i \partial_t \nu$
$\sigma^{\mu\nu} = 2p^{\sigma(\mu} \nabla_{\sigma} u^{\nu)} - \frac{2}{d} p^{\mu\nu} \nabla_\mu u^\nu$	$\sigma^{ij} = 2\nabla^{(i} \bar{u}^{j)} - \frac{2}{d} g^{ij} \nabla_i \bar{u}^j - 2a^{(i} \partial_t \bar{u}^{j)} + \frac{2}{d} g^{ij} a_i \partial_t \bar{u}^j$ $-(e^\Phi - a_i \bar{u}^i) (\partial_t g^{ij} + \frac{1}{d} g^{ij} g_{kl} \partial_t g^{kl})$
Parity Odd—Odd Dimensions ($d = 2n - 1$)	
$l_{(r)}^\mu _{r=0}^{n-1} \equiv *[\beta^{\wedge r} \wedge \omega^{\wedge(n-r-1)}]^\mu$	$l_{(r)}^i _{r=0}^{n-1} \equiv *[\beta^{\wedge r} \wedge \omega^{\wedge(n-r-1)}]^i$
Parity Odd—Even Dimensions ($d = 2n$)	
$l_{(r)}^\mu _{r=0}^n \equiv *[\beta^{\wedge r} \wedge \omega^{\wedge(n-r)}]^\mu$	$l_{(r)}^\mu _{r=0}^n \equiv *[\beta^{\wedge r} \wedge \omega^{\wedge(n-r)}]^\mu$
$l_{(r)}^{\mu\nu} \partial_\nu \vartheta, l_{(r)}^{\mu\nu} \partial_\nu \varpi, l_{(r)}^{\mu\nu} \partial_\nu \nu$	$l_{(r)}^{ij} (\partial_j \vartheta - a_i \partial_t \vartheta), l_{(r)}^{ij} (\partial_j \varpi - a_i \partial_t \varpi), l_{(r)}^{ij} (\partial_j \nu - a_i \partial_t \nu)$
$l_{(r)}^{\mu\nu} (\mathcal{F}_{\nu\rho} u^\rho - \vartheta \partial_\nu \nu)$	$l_{(r)}^{ij} (e_j + \beta_{jk} \bar{u}^k - \vartheta \partial_j \nu + \vartheta a_j \partial_t \nu)$
$l_{(r)}^{\rho(\mu} \sigma^{\nu)\sigma} p_{\rho\sigma}$	$l_{(r)}^{k(i} \sigma^{j)l} g_{kl}$
where, $l_{(r)}^\mu _{r=0}^{n-1} \equiv *[\beta^{\wedge r} \wedge \omega^{\wedge(n-r-1)}]^\mu$	$l_{(r)}^{\mu\nu} _{r=0}^{n-1} \equiv *[\beta^{\wedge r} \wedge \omega^{\wedge(n-r-1)}]^{ij}$

$$W_{\text{anom}}^{eqb} = \int_{\mathcal{M}_{(2n-1)}} n C^{(2n)} \nu_o A \wedge \left((dA)^{\wedge(n-1)} + \frac{1}{2} (n-1) \nu_o \tilde{\vartheta} da \wedge (dA)^{\wedge(n-2)} \right). \quad (133)$$

Above we have left more than one powers of da , as they do not contribute in torsionless configurations. Varying it we can find the anomalous contribution to Galilean currents; only nontrivial contributions are given by,

$$j_{q,\text{anom}}^i = n(n+1) C^{(2n)} \vartheta_o \nu_o * [(dA)^{\wedge(n-1)}]^i, \\ j_{e,\text{anom}}^i = \frac{1}{2} n(n+1) C^{(2n)} \vartheta_o^2 \nu_o^2 * [(dA)^{\wedge(n-1)}]^i. \quad (134)$$

When generating constitutive relations of a Galilean fluid using equilibrium partition function, above results are naturally written in equilibrium hydrodynamic frame. Interestingly mass current does not get any anomalous correction, hence these results are automatically written in mass frame as well. Correspondingly the constraints of parity-odd sector in odd spatial dimensions ($d = 2n - 1$) Eq. (52) modify to include contribution from anomalies,

$$\tilde{\omega}_{e(n-1)} = \vartheta n \left(\vartheta C_{1,(n-1)} + \frac{E + P - \vartheta \varpi R}{R} C_{2,(n-1)} - \vartheta \nu C_{2,(n)} + \frac{1}{2} (n+1) \vartheta \nu^2 C^{(2n)} \right), \\ \tilde{\omega}_{q(n-1)} = \vartheta n \left(\frac{Q}{R} C_{2,(n-1)} - C_{2,(n)} + (n+1) \nu C^{(2n)} \right). \quad (135)$$

Note that only $r = n - 1$ component of Eq. (52) is modified, while other constraints remain unchanged.

C. Anomalous entropy current

In this section, we shall try to get the anomalous contribution to constitutive relations found in the last subsection, using second law constraint. In the presence of anomaly, the canonical entropy current divergence Eq. (45) will get modified to,

$$\vartheta \nabla_M J_{s(\text{can})}^M = -\Pi^{MN} \nabla_M u_N - \frac{1}{\vartheta} \mathcal{E}^M \partial_M \vartheta + \Upsilon^M (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu) + \vartheta \nu (n+1) C^{(2n)} * [\bar{V}_{(T)} \wedge \mathcal{F}^{\wedge n}]. \quad (136)$$

Using leading order parity-odd ($d = 2n - 1$) constitutive relations Eq. (47) we can evaluate it to get,

$$\vartheta \nabla_M J_{s(\text{can})}^M = \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^M \left[-\tilde{\omega}_{e(r)} \frac{1}{\vartheta} \partial_M \vartheta + \tilde{\omega}_{q(r)} (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu) \right] - \vartheta \nu n (n+1) C^{(2n)} l_{(n-1)}^M \mathcal{F}_{MN} u^N. \quad (137)$$

Clearly it will modify the constraints Eq. (135) only for $r = n - 1$. Plugging in the expression for $\nabla_M \Upsilon_s^M$ from Sec. III C 2, we will reproduce expression for $\tilde{\omega}_{q(n-1)}$ in Eq. (135), and get the following differential equations for $\tilde{\omega}_{\epsilon(n-1)}$,

$$\begin{aligned}\tilde{\omega}_{\epsilon(n-1)} &= \vartheta n \left(\vartheta \frac{\partial}{\partial \vartheta} \tilde{\omega}_{s(n-1)} + \frac{E + P - \vartheta \varpi R}{R} C_{2,(n-1)} - \vartheta \nu C_{2,(n)} \right), \\ \frac{\partial}{\partial \varpi} \tilde{\omega}_{s(n-1)} &= 0, \quad \frac{\partial}{\partial \nu} \tilde{\omega}_{s(n-1)} = n(n+1) \vartheta^2 \nu C^{(2n)}.\end{aligned}\tag{138}$$

The last equation will imply,

$$\tilde{\omega}_{s(n-1)} = \frac{1}{2} n(n+1) \vartheta \nu^2 C^{(2n)} + f(\vartheta), \quad \frac{\partial}{\partial \vartheta} \tilde{\omega}_{s(n-1)} = \frac{1}{2} n(n+1) \nu^2 C^{(2n)} + C_{1,(n-1)}(\vartheta).\tag{139}$$

This gives the remaining $\tilde{\omega}_{\epsilon(n-1)}$ constraint in Eq. (135), except that $C_{1,(n-1)}$ is an arbitrary function of ϑ similar to what we saw in Sec. III C 2. This can be remedied by putting in torsion to relax $\mathcal{H}_{MN} = 0$ condition, as we shall present in Appendix A.

VII. DISCUSSION

In this work we have proposed an innovative and interesting approach to construct the equilibrium partition function and constitutive relations of a Galilean fluid in arbitrary dimensions, starting from a relativistic system, namely *null fluid*. The basic idea of this construction has already been presented in a previous paper [26]; here we have generalized it to include a global (anomalous) $U(1)$ current. The beauty and importance of our approach lies in the construction of *null fluids*. We have showed that the symmetries and background field content of a Galilean theory is exactly captured by a theory defined on null background. This motivates us to define a theory of hydrodynamics on null backgrounds (i.e. null fluid) from scratch, and use it to derive constitutive relations of a Galilean fluid.

Although, the main aim of this construction has been to write down leading order constitutive relations of a Galilean fluid (in presence of anomaly), but in the process we learned that null fluids can be considered as a robust stage to study properties of the most generic Galilean fluid. We found an exact one to one correspondence (not corrected order by order in derivatives) between all aspects of a null fluid and a Galilean fluid, but more than that, the actual map of this correspondence is essentially trivial. Our approach has been to study the null fluid itself as an independent theory, and later exploit the triviality of this map to say something useful about the Galilean fluids.

The triviality of this map is an important feature. Past works in this direction, including our own in [15], have at best found a mapping between usual relativistic fluids and (a subset of) Galilean fluids, which has to be corrected order by order in the derivative expansion (not to forget the

faulty thermodynamics it endows to the Galilean fluid), and not much useful could be said thereof. Note that in the current work however, our mapping is exact to all orders in the derivative expansion, which enables us to directly use much sophisticated relativistic machinery to study Galilean fluids and hence is very interesting in realistic scenarios.

We also found that null backgrounds allow us to introduce $U(1)$ anomalies in an odd-dimensional null theory (i.e. an even dimensional Galilean theory) and forbid them in an even dimensional null theory (i.e. an odd dimensional Galilean theory). This is in sharp contrast with the usual relativistic results, where it is the other way round. However from the perspective of Galilean fluids it is pretty natural. A generalization of this construction to include non-Abelian and gravitational anomalies will be presented in a companion paper [18].

Apart from these anomalous terms, we also have other parity-odd terms in the constitutive relations, for both, even and odd dimensional Galilean fluid. The study of parity-odd nonrelativistic hydrodynamics has become a very fascinating topic in recent years. Fluid consisting of chiral molecules breaks parity at the microscopic level. This kind of fluid plays an important role in many biochemical processes, where only the molecule with right chirality can fit into a protein. Therefore to model such a fluid, we would be forced to add parity-odd terms in the constitutive relations. Our construction gives a consistent way to obtain the possible parity-odd terms in a Galilean fluid, at any desired order in the derivative expansion. It would be very interesting and important to understand the effect of these terms in some practical examples.

Finally, we would like to make some comments on physical aspects of the null fluid. Although we construct the null fluid dynamics and show that it is in one to one correspondence to a lower dimensional Galilean fluid; the physical significance of this null fluid itself is not yet clear. The presence of an extra background field, i.e. the null Killing vector, has allowed us to introduce a set of new transport coefficients (e.g. R) in the null fluid constitutive relations, as compared to a usual relativistic fluid. The

physical meaning of these new transport coefficients, which come coupled to the null vector, becomes clear only once we identify the dynamics of null fluid with that of a Galilean fluid living in one lower dimension. Hence, at this stage, the correct physical interpretation of the null fluid appears to be that it is a particular embedding of the Galilean fluid in a spacetime with one higher dimension. This approach is more in lines with the axiomatic approach of defining a Galilean fluid, but has the benefit that we have all the well-developed machinery of relativistic physics at our disposal.

ACKNOWLEDGMENTS

We would like to thank Felix Haehl for various helpful discussions. The work of N.B. is supported by DST Ramanujan Fellowship. A.J. would like to thank Durham Doctoral Scholarship for financial support, and hospitality of IISER Bhopal where part of this project was done. N.B. and S.D. would like to thank the people of India for their generous support to basic science research.

APPENDIX A: MINIMAL TORSION MODEL

In the main text we alluded to more that one instance that entropy current fails to capture all the constraints a null/Galilean fluid should follow on torsionless backgrounds. In this appendix we provide heuristic arguments for consistency between entropy current and equilibrium partition function constraints in parity-odd sector by introducing a “minimal torsion” in the background. These backgrounds lead to the twistless torsional Newton-Cartan (TTNC) backgrounds after null reduction [27]. We introduce a torsion in the connection, $\Gamma^R_{[MN]} = -\frac{1}{2}u^R\mathcal{H}_{MN}$, so the full connection becomes,

$$\Gamma^R_{MS} = \frac{1}{2}G^{RN}(\partial_M G_{NS} + \partial_S G_{NM} - \partial_N G_{MS}) - \frac{1}{2}(u^R\mathcal{H}_{MS} - u_M\mathcal{H}_S^R - u_S\mathcal{H}_M^R). \quad (A1)$$

One can check that with this connection, $\nabla_M V^N = 0$ does not require \mathcal{H}_{MN} to be zero. In fact, for our purposes it suffices to choose $\mathcal{H} = dV = V \wedge X$ where X^M is a projected vector such that $dX = 0$. We call this *minimal torsion*. For spinless theories, constitutive relations remain the same except $\nabla_M \rightarrow \underline{\nabla}_M = \nabla_M - X_M$. We can define an entropy current as before, whose canonical part will now have divergence,

$$\begin{aligned} \vartheta \underline{\nabla}_M J^M_{s(\text{can})} &= -\Pi^{MN}\nabla_M u_N + \mathcal{E}^M \left(X_M - \frac{1}{\vartheta} \partial_M \vartheta \right) \\ &+ \Upsilon^M (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu). \end{aligned} \quad (A2)$$

We will now write the most generic constitutive relations and entropy current corrections in presence of minimal torsion, and compute constraints on hydrodynamic transport. In the following we only consider the parity-odd sector; in the parity-even sector we did not find any discrepancy between entropy current and equilibrium partition function to start with, and moreover calculation with torsion turns out to be trivially equivalent to what was done without torsion.

Odd dimensions ($d = 2n - 1$): In odd dimensions introduction of X^M does not lead to any new data; hence constitutive relations Eq. (47) do not modify, neither does the entropy current correction Eq. (61). However divergence of entropy current does modify,

$$\begin{aligned} -\nabla_M \Upsilon_s^M &= l_{(0)}^M n \left[-\partial_M \tilde{\omega}_{s(0)} + C_{2,(1)} \vartheta \nu \left(X_M - \frac{1}{\vartheta} \partial_M \vartheta \right) - C_{2,(1)} (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu) \right] \\ &+ \sum_{r=1}^{n-1} \binom{n-1}{r} l_{(r)}^M n \left[-\partial_M \tilde{\omega}_{s(r)} + \tilde{\omega}_{s(r)} \frac{1}{\vartheta} \partial_M \vartheta + \left(\frac{\varrho}{R} C_{2,(r)} - C_{2,(r+1)} \right) (\mathcal{F}_{MN} u^N - \vartheta \partial_M \nu) \right. \\ &\left. + \left(\tilde{\omega}_{s(r)} + \frac{E + P - \vartheta \varpi R}{R} C_{2,(r)} - \vartheta \nu C_{2,(r+1)} \right) \left(X_M - \frac{1}{\vartheta} \partial_M \vartheta \right) \right]. \end{aligned} \quad (A3)$$

Plugging in the canonical part of entropy current Eq. (A2) and demanding $\underline{\nabla}_M J_s^M \geq 0$, we will find a consistency condition on entropy current,

$$\begin{aligned} \frac{\partial}{\partial \vartheta} \tilde{\omega}_{s(r)} &= \frac{\tilde{\omega}_{s(r)}}{\vartheta}, & \frac{\partial}{\partial \varpi} \tilde{\omega}_{s(r)} &= 0, \\ \frac{\partial}{\partial \nu} \tilde{\omega}_{s(r)} &= 0 \Rightarrow \tilde{\omega}_{s(r)} &= C_{1,(r)} \vartheta, \end{aligned} \quad (A4)$$

where $C_{1,(r)}$ is a constant. This is the missing constraint, which we did not find through entropy current in absence of torsion. Using this we can check that we get all the constraints Eq. (52) which we got from equilibrium partition function. One can now take a torsionless limit, which is trivial as there is no X_M dependence in the constitutive relations or entropy current.

Even dimensions ($d = 2n$): The even dimensional case is more interesting. Constitutive relations Eq. (48) modifies as,

$$\begin{aligned}\tilde{\Pi}_{(n)}^{MN} &= -P^{MN} \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{(r)} l_{(r)} - \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\eta}_{(r)} l_{(r)}^{R(M} \sigma^{N)}{}_R, \\ \tilde{\mathcal{E}}_{(n)}^M &= \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} [\tilde{\lambda}_{\epsilon\varpi(r)} \partial_N \varpi + \tilde{\lambda}_{\epsilon\nu(r,s)} \partial_N \nu + \tilde{\kappa}_{\epsilon(r)} \partial_N \vartheta + \tilde{\lambda}_{\epsilon X(r)} X_N + \tilde{\sigma}_{\epsilon(r)} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)], \\ \tilde{\Upsilon}_{(n)}^M &= \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} [\tilde{\lambda}_{q\varpi(r)} \partial_N \varpi + \tilde{\lambda}_{q\nu(r)} \partial_N \nu + \tilde{\kappa}_{q(r)} \partial_N \vartheta + \tilde{\lambda}_{qX(r)} X_N + \tilde{\sigma}_{q(r)} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu)].\end{aligned}\quad (\text{A5})$$

On the other hand, most generic entropy current correction (omitting terms that will give pure derivative terms in the divergence) will be given as,

$$\begin{aligned}\Upsilon_s^M &= \sum_{r=0}^n \binom{n}{r} \mathcal{S}_{2,(r)} \star [u \wedge \hat{\mathcal{F}}^{\wedge r} \wedge \hat{\Omega}^{\wedge(n-r-1)}]^M \\ &\quad + \vartheta n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left[\tilde{\lambda}_{s\varpi(r)} \partial_N \varpi + \tilde{\lambda}_{s\nu(r)} \partial_N \nu + \tilde{\lambda}_{s\vartheta(r)} \partial_N \vartheta + \tilde{\lambda}_{sX(r)} \left(X_N - \frac{1}{\vartheta} \partial_N \vartheta \right) \right].\end{aligned}\quad (\text{A6})$$

Its divergence can be computed to be,

$$\begin{aligned}\nabla_M \Upsilon_s^M &= - \sum_{r=0}^n \binom{n}{r} l_{(r)} u^M \partial_M \mathcal{S}_{2,(r)} \\ &\quad + n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left[\left(\frac{Q}{R} \partial_M \mathcal{S}_{2,(r)} - \partial_M \mathcal{S}_{2,(r+1)} \right) (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) \right. \\ &\quad + \left(I_{(r)M} \vartheta + \frac{E + P - \vartheta \varpi R}{R} \partial_M \mathcal{S}_{2,(r)} - \vartheta \nu \partial_M \mathcal{S}_{2,(r+1)} \right) \left(X_N - \frac{1}{\vartheta} \partial_N \vartheta \right) \vartheta \left(\frac{\partial}{\partial \vartheta} \tilde{\lambda}_{s\varpi(r)} - \frac{\partial}{\partial \varpi} \tilde{\lambda}_{s\vartheta(r)} \right) \partial_M \vartheta \partial_N \varpi \\ &\quad \left. + \vartheta \left(\frac{\partial}{\partial \nu} \tilde{\lambda}_{s\vartheta(r)} - \frac{\partial}{\partial \vartheta} \tilde{\lambda}_{s\nu(r)} \right) \partial_M \nu \partial_N \vartheta + \vartheta \left(\frac{\partial}{\partial \varpi} \tilde{\lambda}_{s\nu(r)} - \frac{\partial}{\partial \nu} \tilde{\lambda}_{s\varpi(r)} \right) \partial_M \varpi \partial_N \nu \right],\end{aligned}\quad (\text{A7})$$

where,

$$I_{(r)} = d\tilde{\lambda}_{sX(r)} + \tilde{\lambda}_{s\varpi(r)} d\varpi + \tilde{\lambda}_{s\nu(r)} d\nu + \tilde{\lambda}_{s\vartheta(r)} d\vartheta. \quad (\text{A8})$$

The divergence of the canonical piece on the other hand is given as:

$$\begin{aligned}\vartheta \nabla_M J_{s(\text{can})}^M &= -\Theta \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{(r)} l_{(r)} - \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{MN} \left[\left\{ \tilde{\lambda}_{\epsilon\varpi(r)} \partial_M \varpi + \tilde{\lambda}_{\epsilon\nu(r)} \partial_M \nu + \left(\tilde{\kappa}_{\epsilon(r)} + \frac{1}{\vartheta} \tilde{\lambda}_{\epsilon X(r)} \right) \partial_M \vartheta \right\} \left(X_N - \frac{1}{\vartheta} \partial_N \vartheta \right) \right. \\ &\quad + \left(\tilde{\sigma}_{\epsilon(r)} - \tilde{\lambda}_{\epsilon X(r)} \right) (\mathcal{F}_{MR} u^R - \vartheta \partial_M \nu) \left(X_N - \frac{1}{\vartheta} \partial_N \vartheta \right) \\ &\quad \left. + \left\{ \tilde{\lambda}_{q\varpi(r)} \partial_M \varpi + \tilde{\lambda}_{q\nu(r)} \partial_M \nu + \left(\tilde{\kappa}_{q(r)} + \frac{1}{\vartheta} \tilde{\lambda}_{qX(r)} \right) \partial_M \vartheta \right\} (\mathcal{F}_{NR} u^R - \vartheta \partial_N \nu) \right].\end{aligned}\quad (\text{A9})$$

Comparing the two we will get three consistency conditions on the entropy current,

$$\frac{\partial}{\partial \vartheta} \tilde{\lambda}_{s\varpi(r)} = \frac{\partial}{\partial \varpi} \tilde{\lambda}_{s\vartheta(r)}, \quad \frac{\partial}{\partial \nu} \tilde{\lambda}_{s\vartheta(r)} = \frac{\partial}{\partial \vartheta} \tilde{\lambda}_{s\nu(r)}, \quad \frac{\partial}{\partial \varpi} \tilde{\lambda}_{s\nu(r)} = \frac{\partial}{\partial \nu} \tilde{\lambda}_{s\varpi(r)}, \quad (\text{A10})$$

which will have most generic solution,

$$\begin{aligned}\lambda_{s\varpi(r)} &= \frac{\partial}{\partial \varpi} f_{1(r)}(\vartheta, \varpi, \nu), \\ \lambda_{s\nu(r)} &= \frac{\partial}{\partial \nu} f_{1(r)}(\vartheta, \varpi, \nu) + \frac{\partial}{\partial \nu} f_{2(r)}(\vartheta, \nu) \\ \lambda_{s\vartheta(r)} &= \frac{\partial}{\partial \vartheta} f_{1(r)}(\vartheta, \varpi, \nu) + \frac{\partial}{\partial \vartheta} f_{2(r)}(\vartheta, \nu) + \frac{\partial}{\partial \vartheta} f_{3(r)}(\vartheta),\end{aligned}\quad (\text{A11})$$

for some arbitrary functions $f_{1(r)}(\vartheta, \varpi, \nu)$, $f_{2(r)}(\varpi, \nu)$, $f_{3(r)}(\nu)$. We define,

$$\mathcal{S}_{1,(r)} = \tilde{\lambda}_{sX(r)} + f_{1(r)} + f_{2(r)} + f_{3(r)}. \quad (\text{A12})$$

One can check that $I_{(r)} = d\mathcal{S}_{1,(r)}$. Demanding entropy current divergence to be non-negative one can check that (in torsionless limit) we get all the equilibrium partition

function constraints, as well as the additional entropy current constraints which we found before.

Hence we have established the agreement of equilibrium partition function and entropy current in arbitrary number of dimensions. These results upon reduction, agree with the entropy current calculation for 2 spatial dimensional fluid in [2], except for a few discrepancies. We provide a detailed comparison with their results in Appendix C.

APPENDIX B: NONCOVARIANT RESULTS

We mentioned in the main text that vector field T defines a reference frame. We can go to *local rest* of one such frame by choosing a basis $x^M = \{x^-, t, x^i\}$ such that $V = \partial_-$ and $T = \partial_t$. This essentially amounts to setting $v^i = 0$ in the Newton-Cartan construction. For example in the equilibrium configuration discussed in Sec. II B, we have studied the system in local rest of frame defined by isometry K^M . Using the same field decomposition as given in Sec. II B, partition function Eq. (99) reduces to:

$$\begin{aligned}\delta W &= \tilde{R} \int \{dx^\mu\} \sqrt{-G} \left[e^{-\Phi} \epsilon_{\text{tot}} \delta \Phi - e^{-2\Phi} (j_\epsilon^i - j_\rho^i e^\Phi B_i - j_q^i e^\Phi A_i) \delta a_i + e^{-\Phi} \frac{1}{2} t^{ij} \delta g_{ij} \right. \\ &\quad \left. + (\rho \delta B_i + e^{-\Phi} j_\rho^i \delta B_i) + (q \delta A_i + e^{-\Phi} j_q^i \delta A_i) \right].\end{aligned}\quad (\text{B1})$$

It is exactly same as Eq. (102), except that the observables are now defined with respect to frame T and are not independent of t . Note that choosing a frame fixes the Milne invariance in the partition function, and hence Milne Ward identity $p^i = j_\rho^i$ goes on-shell. Under Milne boost background fields transform as [cf. Eq. (91)],

$$B_i \rightarrow B_i - e^{-\Phi} \frac{1}{2} \tilde{\psi}^2, \quad B_i \rightarrow B_i + \tilde{\psi}_i. \quad (\text{B2})$$

On the other hand various densities and currents transform as [cf. Eq. (97)],

$$\begin{aligned}\rho &\rightarrow \rho, \quad j_\rho^i \rightarrow j_\rho^i - \rho \tilde{\psi}^i, \quad t^{ij} \rightarrow t^{ij} - 2j_\rho^i \tilde{\psi}^j + \rho \tilde{\psi}^2, \\ \epsilon_{\text{tot}} &\rightarrow \epsilon_{\text{tot}} - j_\rho^i \tilde{\psi}_i + \frac{1}{2} \rho \tilde{\psi}^2, \\ j_\epsilon^i &\rightarrow (j_\epsilon^i - \epsilon_{\text{tot}} \tilde{\psi}^i) - (t^{ij} - \tilde{\psi}^i j_\rho^j) \tilde{\psi}_j + \frac{1}{2} \tilde{\psi}^2 (j_\rho^i - \rho \tilde{\psi}^i), \\ q &\rightarrow q, \quad j_q^i \rightarrow j_q^i - q \tilde{\psi}^i.\end{aligned}\quad (\text{B3})$$

The conservation equations (101) in noncovariant basis becomes,

$$\begin{aligned}\frac{1}{\sqrt{g}} \partial_t (\sqrt{g} q_{\text{nc}}) + \nabla_i (e^{-\Phi} j_q^i) &= 0, \\ \frac{1}{\sqrt{g}} \partial_t (\sqrt{g} \rho_{\text{nc}}) + \nabla_i (e^{-\Phi} j_\rho^i) &= 0, \\ \frac{1}{\sqrt{g}} \partial_t (\sqrt{g} \epsilon_{\text{tot,nc}}) + \nabla_i (e^{-\Phi} j_\epsilon^i) &= -\frac{1}{2} t^{ij} \partial_t g_{ij} + e^{-\Phi} e_i j_q^i - e^{-\Phi} \alpha_{(T)i} j_\rho^i \\ \frac{1}{\sqrt{g}} \partial_t (\sqrt{g} p_{\text{nc},i}) + \nabla_j (e^{-\Phi} t^j_i) &= -\frac{1}{2} e^{-\Phi} a_i t^{jk} \partial_t g_{jk} + e^{-\Phi} (q e_i + \beta_{ij} j_q^j) + e^{-\Phi} (-\rho \alpha_{(T)i} + \omega_{(T)ij} j_\rho^j),\end{aligned}\quad (\text{B4})$$

where noncovariant densities gets a contribution from temporal curvature,

$$\begin{aligned}\rho_{\text{nc}} &= \rho - e^{-\Phi} j_\rho^i a_i, & \epsilon_{\text{tot,nc}} &= \epsilon_{\text{tot}} - e^{-\Phi} j_\epsilon^i a_i, \\ q_{\text{nc}} &= q - e^{-\Phi} j_q^i a_i, & p_{\text{nc}}^i &= j_\rho^i - e^{-\Phi} t^{ij} a_j.\end{aligned}\quad (\text{B5})$$

These are just the usual Galilean conservation equations, generalized to curved space-time. Constitutive relations of a Galilean fluid written in mass frame can be found as [cf. Eq. (110)],

$$\begin{aligned}j_\rho^i &= p^i = R\bar{u}^i, & t^{ij} &= R\bar{u}^i \bar{u}^j + P g^{ij} + \pi^{ij}, & j_q^i &= Q\bar{u}^i + \varsigma_q^i, & \epsilon_{\text{tot}} &= E + \frac{1}{2} R\bar{u}^2, \\ j_\epsilon^i &= \left(E + P + \frac{1}{2} R\bar{u}^2\right) \bar{u}^i + \varsigma_\epsilon^i + \pi^{ij} \bar{u}_j \cdot \hat{s} = S - \Upsilon_{s-}, & j_s^i &= S\bar{u}^i + \frac{1}{g} \varsigma_\epsilon^i - \nu \varsigma_q^i + \Upsilon_s^i - \Upsilon_{s-} \bar{u}^i.\end{aligned}\quad (\text{B6})$$

Note that $\bar{u}^i = u^i$, $\bar{u}_i = g_{ij} \bar{u}^j$ and $\bar{u}^2 = \bar{u}^i \bar{u}_i$. They follow conservation laws Eq. (B4). Finally we can explicitly obtain the constitutive relations up to leading order in derivative expansion for by reducing results of Sec. V in mentioned basis. Reduction of various data down to noncovariant basis is given in Table III. In the following we present results for a special case when time is flat ($a_i = \Phi = 0$), space is time independent ($\partial_t g_{ij} = 0$) and reference frame is inertial $\alpha_i = \omega_{ij} = 0$ for simplicity.

Odd spatial dimensions: We first present constitutive relations for a fluid living in odd spatial dimensions ($d = 2n - 1$),

$$\begin{aligned}\pi^{ij} &= -\eta \sigma^{ij} - g^{ij} \zeta \nabla_k \bar{u}^k, \\ \varsigma_\epsilon^i &= \kappa_\epsilon \nabla^i \vartheta + \vartheta \kappa_q (e^i + \beta^{ij} \bar{u}_j - \vartheta \nabla^i \nu) + \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{\epsilon(r)} l_{(r)}^i, \\ \varsigma_q^i &= \kappa_q \nabla^i \vartheta + \sigma_q (e^i + \beta^{ij} \bar{u}_j - \vartheta \nabla^i \nu) + \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\omega}_{q(r)} l_{(r)}^i.\end{aligned}\quad (\text{B7})$$

In the special case of 3 spatial dimensions we will get the well-known results [1],

$$\begin{aligned}\pi^{ij} &= -\eta \sigma^{ij} - \zeta g^{ij} \nabla_k \bar{u}^k, \\ \varsigma_\epsilon^i &= \kappa_\epsilon \nabla^i \vartheta + \sigma_\epsilon (e^i + (\bar{u} \times B)^i - \vartheta \nabla^i \nu) + \tilde{\omega}_{\epsilon(0)} \omega^i + \tilde{\omega}_{\epsilon(1)} B^i, \\ \varsigma_q^i &= \kappa_q \nabla^i \vartheta + \sigma_q (e^i + (\bar{u} \times B)^i - \vartheta \nabla^i \nu) + \tilde{\omega}_{q(0)} \omega^i + \tilde{\omega}_{q(1)} B^i,\end{aligned}\quad (\text{B8})$$

where $B^i = \frac{1}{2} \epsilon^{ijk} \beta_{jk}$, $\omega^i = \epsilon^{ijk} \partial_j v_k$, $(v \times B)^i = \epsilon^{ijk} v_j B_k = \beta^{ij} v_j$.

Even spatial dimensions: Similarly in even spatial dimensions ($d = 2n$) we can get,

$$\begin{aligned}\pi^{ij} &= -\eta \sigma^{ij} - \sum_{r=0}^{n-1} \binom{n-1}{r} \tilde{\eta}_{(r)} l_{(r)}^{k(i} \sigma^{j)l} g_{kl} - g^{ij} \left(\zeta \nabla_i \bar{u}^i + \sum_{r=0}^n \binom{n}{r} \tilde{\zeta}_{(r)} l_{(r)} \right), \\ \varsigma_\epsilon^i &= \left(g^{ij} \kappa_\epsilon + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{ij} \tilde{\kappa}_{\epsilon(r)} \right) \partial_j \vartheta + \vartheta \left(g^{ij} \kappa_q - \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{ij} \tilde{\kappa}_{q(r)} \right) (e_j + \beta_{jk} \bar{u}^k - \vartheta \partial_j \nu) \\ &\quad + \vartheta n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{ij} \left(\vartheta \partial_j \mathcal{S}_{1,(r)} + \frac{E + P - \vartheta \varpi R}{R} \partial_j \mathcal{S}_{2,(r)} - \vartheta \nu \partial_j \mathcal{S}_{2,(r+1)} \right), \\ \varsigma_q^i &= \left(g^{ij} \kappa_q + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{ij} \tilde{\kappa}_{q(r)} \right) \partial_j \vartheta + \left(g^{ij} \sigma_q + \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{ij} \tilde{\sigma}_{q(r)} \right) (e_j + \beta_{jk} \bar{u}^k - \vartheta \partial_j \nu) \\ &\quad + \vartheta n \sum_{r=0}^{n-1} \binom{n-1}{r} l_{(r)}^{ij} \left(\frac{Q}{R} \partial_j \mathcal{S}_{2,(r)} - \partial_j \mathcal{S}_{2,(r+1)} \right).\end{aligned}\quad (\text{B9})$$

These results for sure are cumbersome, but take a cleaner form in 2 spatial dimensions,

$$\begin{aligned}
\pi^{ij} &= -\eta\sigma^{ij} - \tilde{\eta}\varepsilon^{k(i}\sigma^{j)l}g_{kl} - g^{ij}\left(\zeta\nabla_k\bar{u}^k + \tilde{\zeta}_\omega\omega + \tilde{\zeta}_B B\right), \\
\varsigma_\epsilon^i &= (g^{ij}\kappa_\epsilon + \varepsilon^{ij}\tilde{\kappa}_\epsilon)\partial_j\vartheta + \vartheta(g^{ij}\kappa_q - \varepsilon^{ij}\tilde{\kappa}_q)\left(e_j + \epsilon_{jk}\bar{u}^k B - \vartheta\partial_j\nu\right) \\
&\quad + \vartheta\varepsilon^{ij}\left(\vartheta\partial_j\mathcal{S}_{1,(0)} + \frac{E+P-\vartheta\varpi R}{R}\partial_j\mathcal{S}_{2,(0)} - \vartheta\nu\partial_j\mathcal{S}_{2,(1)}\right), \\
\varsigma_q^i &= (g^{ij}\kappa_q + \epsilon^{ij}\tilde{\kappa}_q)\partial_j\vartheta + (g^{ij}\sigma_q + \epsilon^{ij}\tilde{\sigma}_q)\left(e_j + \epsilon_{jk}\bar{u}^k B - \vartheta\partial_j\nu\right) \\
&\quad + \vartheta\varepsilon^{ij}\left(\frac{Q}{R}\partial_j\mathcal{S}_{2,(0)} - \partial_j\mathcal{S}_{2,(1)}\right),
\end{aligned} \tag{B10}$$

where $B = \frac{1}{2}\varepsilon^{ij}\beta_{ij}$, $\omega = \varepsilon^{ij}\partial_i v_j$, $\varepsilon^{ij}v_j B = \beta^{ij}v_j$.

Under the assumption of flat time, time independent space and reference frame being inertial, these constitutive relations follow very simplified and familiar conservation laws,

$$\begin{aligned}
\partial_t Q + \nabla_i j_q^i &= 0, & \partial_t R + \nabla_i(Rv^i) &= 0, \\
\partial_t\left(E + \frac{1}{2}Rv^i v_i\right) + \nabla_i j_\epsilon^i &= j_q^i e_i, & \partial_t(Rv^i) + \nabla_j t^{ji} &= Qe^i - j_{qj}\beta^{ji}.
\end{aligned} \tag{B11}$$

The last term can be seen as $(j_q \times B)^i$ or $\varepsilon^{ij}j_{qj}B$ depending on the number of dimensions.

APPENDIX C: COMPARISON WITH GERACIE *ET AL.* [2]

Our null fluid construction is computationally similar to [2], but has a different essence to it. Authors in [2] considered an extended $(d+2)$ -dim representation of the Galilean group, and realized it with the help of a $(d+2)$ -dim flat space. They then defined an extended Vielbein which connects this $(d+2)$ -dim space to $(d+1)$ -dim Newton-Cartan manifold, and used this formalism to write Ward identities and constitutive relations of the Galilean fluid in a covariant manner. In this work however, we do hydrodynamics on the $(d+2)$ -dim curved manifold (null background) to start with, and later perform light cone reduction to get a Galilean fluid. As we mentioned, fluid on null background (null fluid) is essentially equivalent to the Galilean fluid, so we can expect computational similarities with the construction of [2].

[2] also studied torsional Galilean fluid in 2 spatial dimensions. We should be able to reproduce their results restricted to torsion-less case. In two spatial dimensions ($d=2$), the hydrodynamic frame invariants in parity-odd sector Eq. (118) becomes,

$$\begin{aligned}
\pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} - \tilde{\eta}\epsilon^{\rho(\mu}\sigma^{\nu)\sigma}p_{\rho\sigma} - p^{\mu\nu}(\zeta\Theta + \tilde{f}_\omega\omega + \tilde{f}_B B), \\
\varsigma_\epsilon^\mu &= \vartheta\sigma_T p^{\mu\nu}(\mathcal{F}_{\nu\rho}v^\rho - \vartheta\partial_\nu\nu) + \kappa_T p^{\mu\nu}\partial_\nu\vartheta - \vartheta\tilde{\sigma}_T\epsilon^{\mu\nu}(\mathcal{F}_{\nu\rho}u^\rho - \vartheta\partial_\nu\nu) + \tilde{\kappa}_T\epsilon^{\mu\nu}\partial_\nu\vartheta \\
&\quad - \tilde{m}\epsilon^{\mu\nu}\mathcal{F}_{\nu\rho}u^\rho + \frac{E+P-\vartheta\varpi R}{R}\vartheta\epsilon^{\mu\nu}\partial_\nu\tilde{n} + \partial_\nu\epsilon^{\mu\nu}\tilde{m}_\epsilon, \\
\varsigma_q^\mu &= \sigma_E p^{\mu\nu}(\mathcal{F}_{\nu\rho}u^\rho - \vartheta\partial_\nu\nu) + \sigma_T p^{\mu\nu}\partial_\nu\vartheta + \tilde{\sigma}_E\epsilon^{\mu\nu}(\mathcal{F}_{\nu\rho}v^\rho - \vartheta\partial_\nu\nu) + \epsilon^{\mu\nu}\tilde{\sigma}_T\partial_\nu\vartheta \\
&\quad + \epsilon^{\mu\nu}\partial_\nu\tilde{m} + \frac{Q}{R}\vartheta\epsilon^{\mu\nu}\partial_\nu\tilde{n},
\end{aligned} \tag{C1}$$

where we have made some redefinitions to make results look similar to [2],

$$\begin{aligned}
\tilde{m} &= -\vartheta\mathcal{S}_{2,(1)}, & \tilde{n} &= \mathcal{S}_{2,(0)}, & \tilde{\sigma}_T &= \tilde{\kappa}_q - \frac{1}{\vartheta}\tilde{m}, & \tilde{m}_\epsilon &= \vartheta^2\mathcal{S}_{1,(0)} + \vartheta\nu\tilde{m}, \\
\tilde{\kappa}_T &= \tilde{\kappa}_\epsilon - \frac{2}{\vartheta}\tilde{m}_\epsilon, & \tilde{\sigma}_E &= \tilde{\sigma}_q, & \sigma_E &= \sigma_q, & \sigma_T &= \kappa_q, & \kappa_T &= \kappa_\epsilon,
\end{aligned} \tag{C2}$$

$$\begin{aligned}\tilde{f}_\omega &= \tilde{\zeta}_\omega = -\left[\vartheta^2 \frac{\partial P}{\partial E} \frac{\partial}{\partial \vartheta} + \frac{\partial P}{\partial R} \frac{\partial}{\partial \varpi} + \frac{\partial P}{\partial Q} \frac{\partial}{\partial \nu}\right] \tilde{n}, \\ \tilde{f}_B &= \tilde{\zeta}_B = \left[\vartheta^2 \frac{\partial P}{\partial E} \frac{\partial}{\partial \vartheta} + \frac{\partial P}{\partial R} \frac{\partial}{\partial \varpi} + \frac{\partial P}{\partial Q} \frac{\partial}{\partial \nu}\right] \frac{\tilde{m}}{\vartheta}.\end{aligned}\quad (\text{C3})$$

Relations Eq. (C1) are same as Eq. (6.64) of [2], except few subtle points.²⁰ We do not have the coefficient $\tilde{\kappa}_Q$ while they do not have \tilde{n} and \tilde{f}_ω . We would like to mention few points which might explain this discrepancy. In the following we use notation as used in their paper.

Just before Eq. (6.52) of [2], authors dropped the $\tilde{\zeta}_\omega \omega u^\mu$ term in the entropy current as it gives rise to a “genuine second order data” $\dot{\omega}$. However one can show that just like magnetic field, there exists an independent combination,

$$\tilde{\zeta}_\omega (\omega u^\mu + \tilde{\alpha}_\nu), \quad (\text{C4})$$

which has composite divergence. Here $\alpha^\nu = u^\mu \nabla_\mu u^\nu$ and $\tilde{V}^\mu = \epsilon^{\mu\nu} V_\nu$ is the duality operation. In other words $\dot{\omega} = u^\mu \partial_\mu \omega$ is not an independent genuine data, and can be decomposed using first order Ward identities,

$$\begin{aligned}\dot{\omega} &= -\omega \Theta - \epsilon^{\mu\nu} \nabla_\mu \alpha_\nu \\ &= \left(\frac{q}{\rho} B - \omega\right) \Theta - \epsilon^{\mu\nu} \nabla_\mu e_\nu + \frac{q}{\rho} \dot{B} - \tilde{\nabla}^\mu \left(\frac{q}{\rho}\right) E_\mu \\ &\quad - \tilde{\nabla}^\mu \left(\frac{\epsilon + p}{\rho}\right) G_\mu - \frac{1}{\rho^2} \tilde{\nabla}^\mu \rho \nabla_\mu p.\end{aligned}\quad (\text{C5})$$

Therefore [2] missed the \tilde{n} and its dependent \tilde{f}_ω coefficients. For other discrepancy we note that they have a term $-\frac{1}{T}(\tilde{\sigma}_G - \tilde{\kappa}_E) \tilde{E}^\mu G_\mu$ in entropy production Eq. (6.55), which implies $\tilde{\sigma}_G = \tilde{\kappa}_E$. This will give rise to another consistency condition, which can be read out directly from Eq. (6.62),

$$\begin{aligned}\tilde{c}_Q + T(\partial_Q \tilde{c}_T - \partial_T \tilde{c}_Q) &= 0 \Rightarrow \tilde{c}_Q \\ &= T[\partial_Q f_1(T, \nu_Q) + \partial_Q f_2(\nu_Q)], \\ \tilde{c}_T &= T \partial_T f_1(T, \nu_Q),\end{aligned}\quad (\text{C6})$$

for some arbitrary functions $f_1(T, \nu_Q)$, $f_2(\nu_Q)$. One can in turn absorb these functions by redefining,

$$\tilde{g} \rightarrow \tilde{g} - f_1 - f_2. \quad (\text{C7})$$

As an effect, in Eq. (6.62) of [2] all the c 's drop out. Consequently by Eq. (6.70) all c dependent coefficients $\tilde{\sigma}_\Phi$, $\tilde{\kappa}_Q$, $\tilde{\kappa}_\Phi$ vanish (or can be absorbed in definition of \tilde{m}_e).

²⁰It was brought to our attention that out of the two discrepancies discussed here, one was fixed in a later version of [2].

Barring these modifications, we find our results to be in exact correspondence with [2] for torsionless fluids.

APPENDIX D: CONVENTIONS OF DIFFERENTIAL FORMS

In this appendix we will recollect some results about differential forms, and will set notations and conventions used throughout this work. An m -rank differential form $\mu^{(m)}$ on $\mathcal{M}_{(d+2)}$, can be written in a coordinate basis as,

$$\mu^{(m)} = \frac{1}{m!} \mu_{M_1 M_2 \dots M_m} dx^{M_1} \wedge dx^{M_2} \wedge \dots \wedge dx^{M_m}, \quad (\text{D1})$$

where μ is a completely antisymmetric tensor. On $\mathcal{M}_{(d+2)}$, volume element is given by a full rank form,

$$\epsilon^{(d+2)} = \frac{1}{(d+2)!} \epsilon_{M_1 M_2 \dots M_{d+2}} dx^{M_1} \wedge dx^{M_2} \wedge \dots \wedge dx^{M_{d+2}}, \quad (\text{D2})$$

where ϵ is the totally antisymmetric Levi-Civita symbol with value $\epsilon_{0,1,2,\dots,d+1} = \sqrt{|G|}$ and $G = \det G_{MN}$. Using it, Hodge dual is defined to be a map from m -rank differential forms to $(d+2-m)$ -rank differential forms,

$$\begin{aligned}\star[\mu^{(m)}] &= \frac{1}{(d+2-m)!} \left(\frac{1}{m!} \mu^{M_1 \dots M_m} \epsilon_{M_1 \dots M_m N_1 \dots N_{d+2-m}} \right) dx^{N_1} \\ &\quad \wedge \dots \wedge dx^{N_{d+2-m}}.\end{aligned}\quad (\text{D3})$$

One can check that $\star\star\mu^{(m)} = \text{sgn}(G)(-)^{m(d-m)}$. For us obviously $\text{sgn}(G) = -1$ due to Minkowski signature of the metric, but we tag along this factor for clarity. The exterior product of a differential form is defined to be,

$$\begin{aligned}d\mu^{(m)} &= \frac{1}{(m+1)!} [(m+1) \partial_{[M_1} \mu_{M_2 \dots M_{m+1}]}] dx^{M_1} \\ &\quad \wedge \dots \wedge dx^{M_{m+1}}.\end{aligned}\quad (\text{D4})$$

Integration of a full rank form is defined as,

$$\begin{aligned}\int_{\mathcal{M}_{(d+2)}} \mu^{(d+2)} &= \text{sgn}(G) \int \{dx^M\} \sqrt{|G|} \star[\mu^{(d+2)}] \\ &= \text{sgn}(G) \int \{dx^M\} \sqrt{|G|} \\ &\quad \times \frac{1}{(d+2)!} \epsilon^{M_1 \dots M_{d+2}} \mu_{M_1 \dots M_{d+2}}.\end{aligned}\quad (\text{D5})$$

Here the raised Levi-Civita symbol has value $\epsilon^{0,1,2,\dots,d+1} = \text{sgn}(G)/\sqrt{|G|}$. Integration of an exact full rank form is given by integration on the boundary,

$$\int_{\mathcal{M}_{(d+2)}} d\mu^{(d+1)} = \int_{\partial\mathcal{M}_{(d+2)}} \mu^{(d+1)}, \quad (\text{D6})$$

where given a unit vector \mathbf{N} normal to boundary, volume element on the boundary is defined as $\iota_{\mathbf{N}}\epsilon^{(d+2)} = \star\mathbf{N}$.

1. Newton-Cartan differential forms

A Newton-Cartan differential form is a differential form on $\mathcal{M}_{(d+2)}$ which does not have a leg along V , i.e. $\iota_V\mu^{(m)}$. Such a form can be expanded as,

$$\mu^{(m)} = \frac{1}{m!} \mu_{\mu_1\mu_2\ldots\mu_m} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_m}. \quad (\text{D7})$$

Volume element of NC manifold is defined as,

$$\epsilon^{(d+1)}_{\downarrow} = \star V = \frac{1}{(d+1)!} (V^M \epsilon_{M\mu_1\ldots\mu_{d+1}}) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{d+1}}. \quad (\text{D8})$$

Since there is a nondegenerate metric on NC manifold we cannot define a Hodge dual. Hodge dual can however be defined if we chose a frame T . We can hence define spatial differential forms with the requirement that they should not have any leg along V and $\bar{V}_{(T)}$. For these forms, indices can be raised and lowered using $p^{\mu\nu}$ and $p_{\mu\nu}$. We can define a spatial volume element,

$$\epsilon^{(d)} = \star[V \wedge \bar{V}] = \frac{1}{d!} (V^M \bar{V}^N \epsilon_{MN\mu_1\ldots\mu_d}) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_d}, \quad (\text{D9})$$

and corresponding to it a Hodge duality operation,

$$\begin{aligned} \star[\mu^{(m)}] &= \star[V \wedge u \wedge \mu^{(m)}] = \frac{1}{(d-m)!} \\ &\times \left(\frac{1}{m!} \mu^{\mu_1\ldots\mu_m} \epsilon_{\mu_1\ldots\mu_m\nu_1\ldots\nu_{d-m}} \right) dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_{d-m}}. \end{aligned} \quad (\text{D10})$$

One can check that $\star\star = -\text{sgn}(G)(-)^{m(d-m)}$.

2. Spatial differential forms

Going to the local rest of frame T used to define the Newton-Cartan spatial forms, we can check that the spatial forms behave covariantly on the spatial slice, i.e. can be expressed as,

$$\mu^{(m)} = \frac{1}{m!} \mu_{i_1i_2\ldots i_m} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_m}. \quad (\text{D11})$$

One can check that the volume element $\epsilon^{(d)}$ defined before is indeed a full rank form on the spatial slice and can be written in this setting as,

$$\epsilon^{(d)} = \frac{1}{d!} (V^M \bar{V}^N \epsilon_{MNi_1\ldots i_d}) dx^{i_1} \wedge \ldots \wedge dx^{i_d}. \quad (\text{D12})$$

The Hodge dual \star associated with it serves as Hodge dual operation on the spatial slice,

$$\begin{aligned} \star[\mu^{(m)}] &= \frac{1}{(d-m)!} \\ &\times \left(\frac{1}{m!} \mu^{i_1\ldots i_m} \epsilon_{i_1\ldots i_m j_1\ldots j_{d-m}} \right) dx^{j_1} \wedge \ldots \wedge dx^{j_{d-m}}. \end{aligned} \quad (\text{D13})$$

Finally a full rank spatial form can be integrated on a spatial slice,

$$\begin{aligned} \int_{\mathcal{M}_{(d)}} \mu^{(d)} &= \text{sgn}(G) \int_{\mathcal{M}_{(d+2)}} e^{\Phi} V \wedge \bar{V} \wedge \mu^{(d)} \\ &= \text{sgn}(g) \int \{dx^{\mu}\} \sqrt{|g|} \star[\mu^{(d)}]. \end{aligned} \quad (\text{D14})$$

Here $g = \det g_{ij} = e^{2\Phi} \gamma = -e^{2\Phi} G$. Since g_{ij} is a spatial metric $\text{sgn}(g) = +1$. Other conventions and notations are the same as the relativistic case.

-
- [1] L. Landau and E. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1959).
 - [2] M. Geracie, K. Prabhu, and M. M. Roberts, Fields and fluids on curved non-relativistic spacetimes, *J. High Energy Phys.* **08** (2015) 042.
 - [3] D. Son, Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry, *Phys. Rev. D* **78**, 046003 (2008).
 - [4] D. T. Son, Newton-Cartan geometry and the quantum hall effect, [arXiv:1306.0638](https://arxiv.org/abs/1306.0638).

- [5] K. Jensen, On the coupling of Galilean-invariant field theories to curved spacetime, [arXiv:1408.6855](https://arxiv.org/abs/1408.6855).
- [6] K. Jensen, Aspects of hot Galilean field theory, *J. High Energy Phys.* **04** (2015) 123.
- [7] K. Jensen and A. Karch, Revisiting non-relativistic limits, *J. High Energy Phys.* **04** (2015) 155.
- [8] R. Banerjee, A. Mitra, and P. Mukherjee, A new formulation of non-relativistic diffeomorphism invariance, *Phys. Lett. B* **737**, 369 (2014).

- [9] R. Banerjee, A. Mitra, and P. Mukherjee, Localization of the Galilean symmetry and dynamical realization of Newton-Cartan geometry, *Classical Quantum Gravity* **32**, 045010 (2015).
- [10] R. Banerjee, A. Mitra, and P. Mukherjee, General algorithm for nonrelativistic diffeomorphism invariance, *Phys. Rev. D* **91**, 084021 (2015).
- [11] C. Duval, G. Burdet, H. Kunzle, and M. Perrin, Bargmann structures and Newton-Cartan theory, *Phys. Rev. D* **31**, 1841 (1985).
- [12] B. Julia and H. Nicolai, Null Killing vector dimensional reduction and Galilean geometrodynamics, *Nucl. Phys.* **B439**, 291 (1995).
- [13] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, Boundary stress-energy tensor and Newton-Cartan geometry in Lifshitz holography, *J. High Energy Phys.* **01** (2014) 057.
- [14] M. Rangamani, S. F. Ross, D. Son, and E. G. Thompson, Conformal non-relativistic hydrodynamics from gravity, *J. High Energy Phys.* **01** (2009) 075.
- [15] N. Banerjee, S. Dutta, A. Jain, and D. Roychowdhury, Entropy current for non-relativistic fluid, *J. High Energy Phys.* **08** (2014) 037.
- [16] J. Hartong, E. Kiritsis, and N. A. Obers, Schrödinger invariance from Lifshitz isometries in holography and field theory, *Phys. Rev. D* **92**, 066003 (2015).
- [17] M. Hassaine and P. Horvathy, Field dependent symmetries of a nonrelativistic fluid model, *Ann. Phys. (N.Y.)* **282**, 218 (2000).
- [18] A. Jain, Galilean anomalies and their effect on hydrodynamics, *Phys. Rev. D* **93**, 065007 (2016).
- [19] K. Jensen, Anomalies for Galilean fields, [arXiv:1412.7750](https://arxiv.org/abs/1412.7750).
- [20] I. Bakas and D. Lüst, Axial anomalies of Lifshitz fermions, *Fortschr. Phys.* **59**, 937 (2011).
- [21] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, and T. Sharma, Constraints on fluid dynamics from equilibrium partition functions, *J. High Energy Phys.* **09** (2012) 046.
- [22] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz, and A. Yarom, Towards Hydrodynamics without an Entropy Current, *Phys. Rev. Lett.* **109**, 101601 (2012).
- [23] F. M. Haehl, R. Loganayagam, and M. Rangamani, The Eightfold Way to Dissipation, *Phys. Rev. Lett.* **114**, 201601 (2015).
- [24] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, A theory of first order dissipative superfluid dynamics, *J. High Energy Phys.* **05** (2014) 147.
- [25] N. Banerjee, S. Dutta, and A. Jain, Higher derivative corrections to charged fluids in 2n dimensions, *J. High Energy Phys.* **05** (2015) 010.
- [26] N. Banerjee, S. Dutta, and A. Jain, On equilibrium partition function for non-relativistic fluid, *Phys. Rev. D* **92**, 081701 (2015).
- [27] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, Torsional Newton-Cartan geometry and Lifshitz holography, *Phys. Rev. D* **89**, 061901 (2014).